Day 12: The Table

Opener

5. Distribute the sheets of decimals at your table. Each sheet lists the decimal equivalent for $0/n, 1/n, \ldots, (n-1)/n$ for different n. Each person will look for these two things on their sheet:

- the number of "unique" cycles of digits, and
- the different lengths of repeating decimals.

We consider $\overline{142857}$ and $\overline{428571}$ to be the same cycle, since they use the same digits in the same order. Share your findings with each other. You might find color helpful.

What? **5**? We want you to do these problems in the order they're presented, despite the weird numbering. Honestly we're not sure why they're in this order, but you might be able to make heads of it.

Important Stuff

1. **a.** Complete Handout #1 for Spencer's chessboards. Try to fill in the check marks row-by-row.

b. How many chessboard colorings are "unique" up to re-orientations?

c. Use the handout to help you determine the number of "unique" colorings, if the color of each square of the chessboard can be independently chosen from a palette of 31 colors.

Spencer believes chess is way more fun on a 2-by-2 board. Spencer is wrong.

3. Today's Problem 1, Problem 9 from Day 2, and Problem 1 from Day 11 all involve coloring four things (squares, beads, or corners) and two color choices. Why aren't all the answers the same?

Problem 1? You mean the first problem, right? No. But this is Problem 3, and it's the third problem, right? Yes. Why are you doing this? I don't know, third base.

6. Revisit Problem 1 from Day 11. This time, think of the rectangles as stickers, so you can't flip them over and can only rotate them.

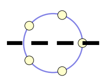
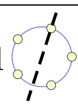
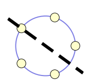
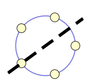
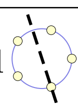
a. How many unique colorings are possible under rotations only?

b. How many unique colorings are possible if you use a palette of 10 colors instead of 2?

2. Trisha is cornering the market on five-bead bracelets. Since she's picked 10 colors for her beads, there are 10^5 possible bracelets if we are unconcerned with re-orientations.

Trisha's 10 colors are: udon know me, schnapps out of it, so many clowns so little time, no more mr. night sky, can't a fjord not to, rice rice baby, tempura-ture rising, pot calling the kettle taupe, machu peach-u, and you're such a buda-pest.

a. Complete this table.

	How many of the 10^5 bracelets have this symmetry?
Rot 0° ↺ (identity)	
Rot 72° ↺	
Rot 144° ↺	
Rot 216° ↺	
Rot 288° ↺	
Refl 	
Refl 	
Refl 	
Refl 	
Refl 	

Is this an obnoxious table-spreading table? Maybe a little, but it's got lots of little coicles.

Fun fact about llamas: every llama's last name is Carson.

Ooh, another fun fact about llamas: Hannibal and his army crossed the Alps on llamas to defeat Rome.

b. How many "unique" bracelets are there? A bracelet is "unique" if it cannot be made from any other bracelet through rotations and reflections. Hint: Your answer should be less than 10^5.

This problem is all about pentagons, man. But did you know Pentagon Man also hates Person Man?

12. If Trisha has k colors for each bead, find a formula in terms of k for the number of "unique" bracelets. Verify that your formula works when $k = 10$ and when $k = 2$.

7. Find the number of six-bead bracelets that are unique up to rotations and reflections, if there are k possible colors for each bead.

Neat Stuff

13. **a.** Find the number of bracelets with 7 equispaced beads that are unique up to rotations and reflections, if there are k possible colors for each bead.

b. 8 beads!

So now we've dealt with necklaces of sizes 5, 6, 7, 8. But what about Schlemiel, Schlimazel, and Hasenpfeffer Incorporated?

10. In these bracelet problems, we've allowed reflections. What if the the bracelets are really regular polygons on stickers, so they can't be turned over? How many unique colorings are possible up to rotations only?

14. Real chessboards have 64 squares arranged in a 8×8 grid. Determine how many colorings are unique up to rotations only, if each square is colored one of 10 colors? k colors?

Finally a real chessboard . . . wait, that square looks like it's the color of tempura, and that one looks like a clown. What a budapest!

9. Find the number of bracelets with n equispaced beads that are unique up to rotations and reflections, if there are k possible colors for each bead. Your answer will depend on whether n is even or odd.

4. If you haven't already done Problem 12 from Day 11 and seen `go.edc.org/gleamingcube`, please do so now. Make sure you can enumerate all 24 rotational symmetries of the cube.

Each animation shows *one* of several possible versions of that type of symmetry. To make the animation more interesting, a cowbell will be hit every time the cube rotates into a symmetry. What? The cowbell player got sick after one too many games of pickleball? Oh, one too many pickle*backs*, I see.

8. Alex wants to color each face of a cube from a palette of k colors. Not considering rotations of the cube, there are k^6 different colorings possible.

a. How many of those k^6 colorings have Amanda #1's 90° rotational symmetry?

b. How many of those k^6 colorings have Amanda #1's 180° rotational symmetry?

c. How many of those k^6 colorings have Amanda #2's 120° rotational symmetry?

d. How many of those k^6 colorings have Amanda #3's 180° rotational symmetry?

e. Use this information to determine the number of colorings of the faces of a cube that are unique up to rotations. Check that your answer matches your previous work for $k = 2$.

Spoiler alert! Do not turn the page! There is a monstrous function at the end of this problem set!

11. Gen stealthily writes down the function

$$f(k) = \frac{1}{24}\left(k^6 + 3k^4 + 12k^3 + 8k^2\right).$$

Calculate $f(2)$ and $f(3)$.

Well, look at that! I, loveable, furry old Grover am the monster at the end of this problem set. And you were so scared.

15. A d6 has the numbers 1 through 6 on it. How many "unique" ways are there to put the numbers on a d6? Here, we mean unique up to rotations.

Thank goodness, the problem numbers are back to normal now. I can't imagine what sort of message it sends to have 14 numbers put in a specific order like that.

16. Ishrat and Max encourage you to go back to Handout #2 from Day 11 and try to "count vertically" instead of horizontally in the case of 31 colors.

Never heard of a d6? It's also known as a *number cube*. The state motto of New Hampshire is "Live Free Or Number Cube". One of the worst James Bond movies was "Number Cube Another Day", wbich was much worse than "Live And Let Number Cube". None of these movies starred Vin Number Cubesel, but Vin has been to Sundance to see some low-budget innumber cube films. The d6 is also a significant part of the Keto Number Cubet. If you're part of a Cutting Crew, you can sing "I Just Number Cubed In Your Arms Tonight". But if you prefer classical music, perhaps Mozart's "Number Cubes Irae" is more your style. Thanks, thanks, you've been a terrific aunumber cubence.

17. Gareth wants to color each *vertex* of a cube from a palette of k colors. How many colorings are unique up to rotations?

18. Jane wants to color each *edge* of a cube from a palette of k colors. How many colorings are unique up to rotations?

19. An element of a group is a *generator* if repeated operation of that element takes you through every element of the group.

- **a.** Find all the generators for $(\mathbb{Z}_{12}, +)$ or explain why there aren't any.
- **b.** Find all the generators for $(\mathbb{U}_{12}, \times)$ or explain why there aren't any.

20. Sometimes $(\mathbb{U}_n, \times)$ has a generator, sometimes it don't.

- **a.** Under what conditions will $\mathbb{U}_n$ have a generator?
- **b.** In terms of n, *how many generators* are there?

Sometimes you feel like a generator, sometimes you don't.

Tough Stuff

21. How many "unique" ways are there to color 3 vertices of an icosahedron white, while coloring all other vertices black?

22. How many non-isomorphic graphs with n vertices are there? And what does this question even mean?