Day 13: The Chain

Important Stuff

1. **a.** What is the decimal expansion for $\frac{1}{39}$?

b. Hey! Watch this: `http://go.edc.org/40cards`. There are ten piles; what pile is card #1 in before each shuffle?

c. Use the piles to find the decimal expansion for $\frac{2}{39}$ and $\frac{13}{39}$.

d. Can you explain why this works?

Yo! Where my opener at! Check the message on the projectors.

2. **a.** What are the six powers of 10 in mod 77?

b. One of the charts from Day 12 included decimal expansions of $n/77$. What do you notice about the decimal expansions for these six fractions:

$$\frac{1}{77}, \frac{10}{77}, \frac{23}{77}, \frac{76}{77}, \frac{67}{77}, \frac{54}{77}$$

Why does that happen?

c. Find all *eleven* solutions of the equation

$$22x = 0 \qquad \text{in mod } 77.$$

This equation says "22 multiplied by what number is a multiple of 77?". Bowen likes 77 because it is $7 \cdot 11$.

d. Find all *some number of* solutions of the equation

$$100x = x \qquad \text{in mod } 77.$$

100 and 99 are both too big for mod 77, so make them smaller! (This was a rejected Carly Rae lyric.)

e. Which $n/77$ have a repeating decimal length of 2?

3. The decimal expansions for the fractions $\frac{1}{81}, \frac{2}{81}, \ldots \frac{80}{81}$ exhibit repeating decimals with lengths 1, 3, and 9.

a. For which n will $n/81$ have repeating decimals of length 1? 3? 9?

b. Find all solutions of the equation

$$1000x = x \qquad \text{in mod } 81.$$

Since 1000 and 999 are too big, you can "reduce" them into mod 81 first. If it's helpful. And it probably is.

4. The decimal expansions for the fractions $\frac{1}{143}, \frac{2}{143}, \ldots \frac{142}{143}$ exhibit repeating decimals with lengths 2 and 6.

a. Find all solutions of the equation

$$100x = x \qquad \text{in mod } 143.$$

This problem brought to you by Brooklyn Nine-Nine, now on NBC! 99 and 143 are both multiples of . . .

. . . sorry, I'm being told there was a one-character typo in the top note.

b. Find all solutions of the equation

$$1000x = x \qquad \text{in mod } 143.$$

. . . that previous note brought to you by Stranger Things, home of Eleven!

5. Complete this table.

	How many integers from 0 to 142 make this equation true in mod 143?
$10^0 x = x$	
$10^1 x = x$	
$10^2 x = x$	
$10^3 x = x$	
$10^4 x = x$	
$10^5 x = x$	

This box presented by the number 143! Now with the flavor of twin primes! Why not use the favorite number of Mr. Rogers today? When you need to say "I love you", say it with 143!

How could you use this table to calculate the number of unique cycles of repeating digits are present in the decimal expansions of $\frac{0}{143}, \frac{1}{143}, \ldots, \frac{142}{143}$?

I think we're supposed to count some symmetries here . . .

6. Use the method above to find the number of unique cycles for some of the sets of decimal expansions that you looked at on Day 12.

7. Change the method above to find the number of unique cycles for some of the decks of cards you've looked at previously.

Remember, if you want to work with a 52-card deck, it's controlled by mod 51, not mod 52.

Review Your Stuff

8. We traditionally set aside part of the last problem set for review. Work as a group at your table to write **one** review question for tomorrow's problem set. Spend **at most 20 minutes** on this. Make sure your question is something that ∗**everyone**∗ at your table can do, and that you expect ∗**everyone**∗ in the class to be able to do. Problems that connect different ideas we've visited are especially welcome. We reserve the right to use, not use, or edit your questions, depending on how much other material we write, whether you've written your question on the approved piece of paper, your group's ability to write a good joke, how good your bribes are, and hundreds of other factors.

Imagine yourself writing an Important Stuff question, that's what we are looking for here. You got this!

Remember that one time at math camp where you wrote a really bad joke for the problem set? No? Good.

If you need any card shuffling animations, see `go.edc.org/2piles` or `go.edc.org/3piles`.

Neat Stuff

9. If b and d are primes, how many units are in each mod (in terms of b and d)?

Mod *bee*?

a. mod b
b. mod bd
c. mod b^2
d. mod b^2d

10. Given n and its prime factorization, how many units are in mod n?

11. Amy wants to corner the made-to-order market for bracelets with three equispaced beads. Determine the number of unique bracelet designs (up to rotations and reflections) if she allows for 2 colors. 3 colors? 4 colors? k colors?

Did anyone see the triple play by the Bees last night? I think that's what this problem is about. Oh, *beads*!

12. If you're hankering for a general formula for the number of unique bracelet designs for n equispaced beads and k colors, then you might enjoy making this table.

I've got a hanker, chief!

	How many bracelets have this symmetry?				
	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$\cdots$
Rot 0 sides ↺					
Rot 1 side ↺					
Rot 2 sides ↺					
Rot 3 sides ↺	(shaded)				
Rot 4 sides ↺	(shaded)	(shaded)			
Rot 5 sides ↺	(shaded)	(shaded)	(shaded)		
$\vdots$	(shaded)	(shaded)	(shaded)	(shaded)	

Here, "Rot p sides ↺" refers to the counterclockwise rotation of the regular polygon by $360° \cdot p/n$.

13. Enumerate the 12 rotational symmetries of a regular tetrahedron.

14. **a.** Bill wants to color each face of a regular tetrahedron from a palette of k colors. How many colorings are unique up to rotations?
b. Jasper wants to color each *vertex* of a regular tetrahedron from a palette of k colors. How many?

What's the official insect of Utah? You would think, but it's actually the aphid.

c. Westley wants to color each *edge* of a regular tetrahedron from a palette of k colors. What now!

Just kidding. Of course it's a bee!

15. Suppose you have a geometric object and a group of symmetries of that object. If Σ means to add up things, then this shtuff and shtuff below is true.

$$\begin{aligned}
\text{\# of ``unique'' colorings of that object up to symmetries} &= \sum_{\text{all colorings}} \frac{1}{\text{size of orbit of that coloring}} \\
&= \sum_{\text{all colorings}} \frac{\text{\# of symmetries of that coloring}}{\text{\# of symmetries in the group}} \\
&= \frac{1}{\text{\# of symmetries in the group}} \sum_{\text{all colorings}} \begin{pmatrix}\text{\# of symmetries} \\ \text{of that coloring}\end{pmatrix} \\
&= \frac{1}{\text{\# of symmetries in the group}} \sum_{\text{all symmetries}} \begin{pmatrix}\text{\# of colorings that} \\ \text{have that symmetry}\end{pmatrix}
\end{aligned}$$

There ain't no reason A and B should be alone
Today, yeah baby, today, yeah baby
I'm on the vertex of glory
And I'm hanging on a corner with you
I'm on the vertex, the vertex, the vertex, the vertex, the vertex, the vertex, the vertex
I'm on the vertex of glory
And I'm hanging on a corner with you! I'm on the corner with you!

What does all of this mean and what does it have to do with "unique" bracelets, chessboards, repeating decimals, and card positions during perfect shuffles? Justify each equal sign above.

"What does it all mean, man?" Be sure to say that using George Carlin's voice, without the seven words please.

Tough Stuff

16. **a.** A d20 is labeled with the numbers 1 through 20, one on each of the faces of a regular icosahedron. How many "unique" d20 options are there?

b. Players would object to a d20 if opposite numbers weren't on opposite faces. The 1 needs to be directly opposite the 20. How many "unique" d20 arrangements are there given this restriction?

What? You'd think you would want the 20 right next to the 1 for maximum suspense.

17. Sicherman dice are a *different* way to populate 2d6 with positive integers, so that the sums of the two d6 matched the usual distribution. Only positive integers are allowed, and repetition is allowed.

2d6 means two six-sided dice.

a. What numbers are on the Sicherman dice?

b. Find all possible "Sicherman-like" dice for the d4, d8, d12, and d20. There may be more than one possible answer, or none at all! Woo hoo ha ha ha.