Day 14: The See

Opener

1. **a.** What are the eight powers of 2 in mod 51?

b. One of the charts from Day 9 included base 2 "decimal" expansions of $n/51$. What do you notice about the base 2 "decimal" expansions for these eight fractions:

$$\frac{1}{51}, \frac{2}{51}, \frac{4}{51}, \frac{8}{51}, \frac{16}{51}, \frac{32}{51}, \frac{13}{51}, \frac{26}{51}$$

Why does that happen?

c. Find all *three* solutions of the equation

$$3x = 0 \qquad \text{in mod } 51.$$

d. Find all *whatever number of* solutions of the equation

$$64x = x \qquad \text{in mod } 51.$$

e. Which $\frac{n}{51}$ have a repeating base 2 "decimal" of length 2?

The See? What sort of weird title is that? You'll have to look back on the titles of this week's problem sets to find a connection. Let us know what you saw.

Also, did you discover the secret message? Set 12's weird problem ordering and sidenotes might help you.

64 is too big, so you might start by making it smaller. Did you know that 525,600 is the same as 64 in mod 51? I hope not, because it's not true.

Important Stuff

2. Remember `http://go.edc.org/40cards` from the last problem set? Cal tells you to investigate card #13. What's going on?

Sit! Sit! Sit! Sit!

3. A 341-card deck undergoes perfect shuffling. We know, it's a lot to take in.

a. What is the smallest integer n so that $2^n - 1$ is a multiple of 341?

b. Can you explain why your answer to part a implies that a 341-card deck restores in 10 shuffles?

c. It would be difficult to directly find the number of unique cycles of the 341-card deck as it undergoes shuffling. So, let's use its 10 "symmetries": shuffle 0 times, shuffle 1 time, . . . , shuffle 9 times! Complete this table to figure this out. Wait, where's the table? Hm, it's around here somewhere.

But does it undergo perfect *Cupid* shuffling? Boy, that song sure says Cupid a lot.

Yesterday it was 143, now it's 341? Crazy. 341 is also a product of two primes, but not twin primes, and in text it means "You Love I". That usually leads to a bad-grammar breakup.

	How many solutions in mod 341?
$2^0x = x$	
$2^1x = x$	
$2^2x = x$	
$2^3x = x$	
$2^4x = x$	
$2^5x = x$	
$2^6x = x$	
$2^7x = x$	
$2^8x = x$	
$2^9x = x$	

The values in this table grow and shrink faster than the speed of that version of Jingle Bells. Did you know 341 is $31 \cdot 11$? Oh hey, that's true!

d. Now, use the method from Day 13 to determine the number of "unique" cycles in a 341-card deck.

The solution is the same as the number of games played by each Premier League team each season. Fine, not helpful? It's also the number of times Daniel sang hallelujah.

4. If you did Problems 6 and 9 from Day 4 or Problems 10 and 11 from Day 5, use the method of the previous problem to verify the number of unique cycles in the 15-card and 21-card decks.

5. How many unique cycles of repeating digits are present in the decimal expansions of $\frac{0}{1507}, \frac{1}{1507}, \ldots, \frac{1506}{1507}$? In case you were wondering,

$$\frac{1}{1507} = 0.\overline{00066357}$$

Stuff Behind the Stuff

6. **a.** Why is doing perfect shuffles (into 2 piles) with 40 cards essentially the same as with 39 cards?

b. Why do perfect shuffles into n piles involve multiplying by n in some mod?

c. How can you determine the minimum number of perfect shuffles required to restore a deck of n cards, without actually performing the shuffles?

This week on Behind the Stuff: Katy Perry's decision to retitle "Left Shark" to the ultimately more popular "Firework".

And without having someone else perform the shuffles . . .

7. Anne and Kate are looking at expressing a whole number and a fraction into different bases. They are comparing Problem 5 from Day 6 and Problem 3 from Day 7. What is the same and different between those two problems? Why?

Like Eat It versus Beat It, they are mostly pretty similar.

8. What happens when a number in base 10 is multiplied by 10? Table 7 wants to encourage everyone (including us) to consider replacing the phrase "shift decimal point" with "every digit changes its place value." Discuss why might this better support students' thinking.

9. What's the formal reasoning behind the method we used to count the number of unique colorings of an object up to a group of symmetries? See Problem 15 from Day 13.

Formal reasoning wears a cummerbund.

10. **a.** The numbers on a deck of cards are like the colors of beads on a necklace. Discuss.

b. Out of all of the possible rearrangements (transformations) of a deck of n cards, the perfect shuffle generates a special group of rearrangements that cycles back to zero perfect shuffles (the identity transformation). Discuss.

Float like a ten of hearts, sting like a bead. *Bead?*

. . . Yes it does!

Your Stuff

Table 1 Monica decides to start a cube necklace business. Her first necklace design features a necklace with four rotating cubes that can move around the necklace. The sides of the cube that the necklace chain goes through do not show or matter. She decides to use her four favorite colors: Bowen Blue, Chartruse, Razzmatazz and Darryl Strawberry. How many unique necklaces can she wear?

Your jokes. *Our jokes.*

Monica's cube necklaces are gleaming!

For all you New Yorkers out there... You're welcome. DAAAAAAARRRYYLLL!

Table 2 Liz loves to play Pinochle. A Pinochle deck has 48 cards, 9 through A, in each suit represented twice. Her brand new deck of cards come in this order:

$$\begin{matrix} 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & \dots \\ \diamondsuit & \diamondsuit & \heartsuit & \heartsuit & \clubsuit & \clubsuit & \spadesuit & \spadesuit & \end{matrix}$$

a. How many perfect shuffles will it take to get the cards back to this specific order? Perfect shuffles into 3 piles?

b. Rob plays double deck Pinochle, which means there are two decks combined like this:

$$\begin{matrix} 9 & 9 & 9 & 9 & 9 & \dots \\ \diamondsuit & \diamondsuit & \diamondsuit & \diamondsuit & \heartsuit & \end{matrix}$$

Rob also likes double-decker buses, upper-decker home runs, retired wide receiver Eric Decker, and tools from a specific company.

how many perfect shuffles will it take to get the cards back to this specific order? Perfect shuffles into 3 piles?

c. Noel likes Jokers and always puts one as the top card. How many perfect shuffles are needed now?

d. With or without Jokers, can you come up with an ordering of the deck that requires fewer perfect shuffles to return to? What if you can also change the deck size?

Table 3 **a.** How could you do Bowen's magic trick with a 10-pile shuffle of 40 cards? Is it possible?

b. How would the answer change if you had a 5-pile shuffle with a 40-card deck? Explain the process, you don't need to explain the exact calculations.

How could you, after all we've been through together!?! Also, did you talk to Table 7?

Table 4 **a.** Do fractions exist in modular arithmetic? Can you reduce them?

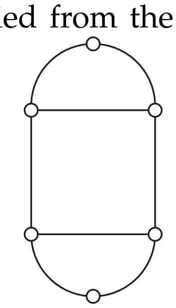
b. Under what conditions can you find solutions to these "mod" equations?

(i) $4x = 1$ in mod 9	(iv) $3x = 5$ in mod 9
(ii) $8x = 2$ in mod 9	(v) $\frac{1}{4} = x$ in mod 9
(iii) $7x = 4$ in mod 9	(vi) $\frac{1}{2} = x$ in mod 51

c. How does your answer to (vi) above relate to card shuffling for a 52-card deck?

Do mother functions exist in modular arithmetic? Please say yes.

Table 5 PCMI's planning committee for 2019–2020 is considering a double-sided reversible name tag so that participants can be identified from the front or the back.



a. PCMI is offering 6 locations for swag tag customization, indicated by the empty circles above. If there are 2 swag tags "Teaching is Lit" and "Mother functions!", how many unique name tag designs are possible up to re-orientations?

Starts on the inside, ya dig?

TI has a song called My Swag.

I'm chasin' that paper, baby, however it come.

b. What if Bowen takes over the swag tag maker and prints out k different swag tags?
c. What swag tags should PCMI use for next summer?

It's a takeover! But never for the front side. Bowen got back.

Table 6 How many shuffles does it take for 85 cards split into 5 piles to return to its original order? What are the unique cycles for this situation?

Table 7 **a.** How do you do perfect shuffles for an arbitrary deck size when you split it into 5 piles? What about n piles?
b. Do Bowen's card trick for 40 cards shuffled into 10 piles in your head!
c. 26 and 1/26 are multiplicative inverses in base 10. Are they multiplicative inverses in base 2?

You can have a pile of 26 cards or a pile of 4 cards, but what do you call a pile of kittens? A MEOW-ntain!

An inverse joke is like a normal joke, but the punchline comes first.

Table 8 Spencer is now coloring a 2×3 chessboard.
a. How many symmetries does an uncolored 2×3 chessboard have?
b. How many unique (up to re-orientations) colorings of the chessboard exist using 2 colors?
c. Using k colors?

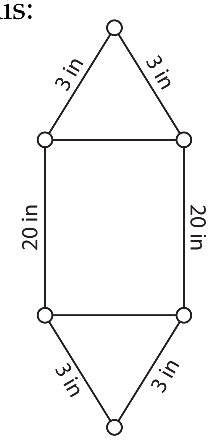
What do baby parabolas drink? Quadratic formula!

Table 9 Let's perfectly shuffle a 36-card deck! Zarina splits the deck into 2 piles while shuffling. Alissa splits the deck into 3 piles. Each person should choose a card number to follow for Zarina's process or Alissa's process. What stays the same? What is different?

1. Bowen was a contestant on *Who Wants to be a Millionaire*; 2. Bowen has been a catalog model; 3. Two of these three statements are false.

Table 10 You're at the build-a-necklace necklace-a-roo (a highly competitive event in Bark Silly, μtah). The necklaces look like this:

This competition seems highly saucy! Bring in Patricia!



You want to predict all of your opponent's designs (uniquely, up to re-orientations). Competitors can only choose from k colors for the beads, but the string lengths have to stay the same. How many unique necklaces do you, the necklace-a-roo master need to worry about?

Table 11 While looking for the next day's problem set in Darryl's room, Bowen finds Darryl's secret stash of triangle-shaped playing cards. He decides he should try a perfect 3-shuffle. Check out `go.edc.org/3piles`.

a. Create a table for the number of perfect 3-shuffles needed to restore different deck sizes.

b. For what deck sizes are the number of perfect 2-shuffles needed the same as the number of 3-shuffles?

Triangle card, triangle card, triangle card hates rectangle card, they have a fight, triangle wins, triangle card.

What's a 4-shuffle like? It's not important. Triangle card.

Table 12 Darryl and Bowen travel to Mirror Land, where only reflections exist. However, upon arrival they soon astonish all of the inhabitants of this land by showing them how to use two consecutive reflections to produce rotations. Given a regular n-gon, figure out how to use any two of its n reflective symmetries to produce any of its n rotational symmetries.

I'm looking at the man in the mirror. He's claiming that we can't rotate... Just take a look at yourself and make that change.

Mirror mirror on the wall, who is the reflectiest of them all?

Oh where oh where has Poly gone? *Poly went to get a cracker. Poly went to Nesia. Poly runs the gamUT.*

More 3-D Stuff

11. Kristen and Jake highly encourage you to go look at those problems involving the cube if you haven't already done them. They're fun! Start at Problem 12 from Day 11 then do Problem 8 from Day 12.

12. PCMI's logo is a regular octahedron. Enumerate the 24 rotational symmetries of a regular octahedron.

But you didn't have to come to Utah
Meet some friends and shuffle cards and then mod some numbers
I guess you've got to leave us though
Now we're just some math camp that you used to know

13. **a.** Jessica wants to color each vertex of a regular octahedron from a palette of k colors. How many colorings are unique up to rotations? Why does this answer look so familiar?

b. Max wants to color each face of a regular octahedron from a palette of k colors. How many colorings are unique up to rotations?

c. Anne wants to color each edge of a regular octahedron from a palette of k colors.

Schtuff and Schtuff

5. Draw a table.

4. Dylan insists in trying to do perfect shuffles on a deck of 52 cards, but splitting it into 10 piles. Define how these shuffles are going to work and figure stuff out.

Dylan also insists that he does not like your girlfriend. No way, no way.

3. Here are six functions.

- $m(x) = 1 - \dfrac{1}{1-x}$
- $o(x) = 1 - x$
- $n(x) = 1 - \dfrac{1}{x}$
- $i(x) = x$
- $c(x) = \dfrac{1}{x}$
- $a(x) = \dfrac{1}{1-x}$

a. Build an operation table for working with these six functions, where the operation is "composition". For example, $n(o(x)) = m(x)$.

b. Which, if any, other tables can match this table by pairing the entries in some way?

Now and then I think of all the times you gave me Neat Stuff
But had me believing it was always something I could do
Yeah I wanna live that way
Reading the dumb jokes you'd play
But now you've got to let us go
And we're leaving from a math camp that you used to know

2. The number of orientations of a d12 and a d20 is the same. Are these isomorphic doyathink?

1. Figure out how to use shuffling to find the *base-10* decimal expansion of $\frac{1}{51}$ along with other fractions.

Don't you forget about us
We'll be alone, shufflin', you know it baby
Bracelets, we'll take them apart
Then put 'em back together in parts, baby
I say $(LA)^{55}$
When you walk on by
Will you call me maybe . . .
(See you again soon.)

No More Stuff