Day 1: Shuf'ling

Welcome to PCMI. We know you'll learn a lot of mathematics here—maybe some new tricks, maybe some new perspectives on things with which you're already familiar. A few things you should know about how the class is organized.

- **Don't worry about answering all the questions.** If you're answering every question, we haven't written the problem sets correctly.

Some of the problems have yet to be solved. Those are the *really* fun ones.

- **Don't worry about getting to a certain problem number.** Some participants have been known to spend the entire session working on one problem (and perhaps a few of its extensions or consequences).
- **Stop and smell the roses.** Getting the correct answer to a question is not a be-all and end-all in this course. How does the question relate to others you've encountered? How do others think about this question?
- **Be excellent to each other.** Believe that you have something to learn from everyone else. Remember that everyone works at a different pace. Give everyone equal opportunity to express themselves. Don't be afraid to ask questions.
- **Teach only if you have to.** You may feel the temptation to teach others in your group. Fight it! We don't mean you should ignore your classmates but give everyone the chance to discover. If you think it's a good time to teach your colleagues about the orbit stabilizer theorem, think again: the problems should lead to the appropriate mathematics rather than requiring it.
- **Each day has its Stuff.** There are problem categories: Important Stuff, Neat Stuff, Tough Stuff. Check out the Opener and the Important Stuff first. All the mathematics that is central to the course can be found and developed there. *That's* why it's Important Stuff. Everything else is just neat or tough. Each problem set is based on what happened the day before.

When you get to Day 3, come back and read this introduction again.

Will you remember? Maybe . . .

Opener

1. Let's watch a video. Don't worry, it's only like 2 minutes long.

Wait *what?* Discuss this with your tablemates and try to figure out what you can about this. After 15 minutes, move on to Problem 2.

What is this I don't even.

Important Stuff

2. Does the perfect shuffle work for other deck sizes? If not, why not? If so, what stays the same and what changes? Distribute the work among your tablemates and discuss.

3. Melanie is thinking of a positive integer, and because she's a math teacher she calls it x. What information would you know about x based on each statement?

a. $3x$ has last digit 4
b. $7x$ has last digit 4
c. $4x$ has last digit 4
d. $5x$ has last digit 4

4. **a.** What number is $9 \cdot 10^1 + 9 \cdot 10^0 + 4 \cdot 10^{-1} + 4 \cdot 10^{-2}$?
b. Noel's favorite number is 802.11_{10}. Write it as a sum of powers of 10.

The little 10 here means the number is in base 10. Bases will generally be given when they seem needed, and we'll try not to be confusing.

5. **a.** What number is $1 \cdot 3^3 + 0 \cdot 3^2 + 2 \cdot 3^1 + 0 \cdot 3^0 + 1 \cdot 3^{-1}$?
b. Brittany's favorite base-3 number is 2110.2_3. Write it as a sum of powers of 3.
c. Convert 2110.2_3 to base 10.

Surely you came to PCMI armed with your favorite number in each base.

6. Write each number as a decimal. Write each number as a decimal. Write each number as a decimal.

These directions either terminate or repeat.

a. $\frac{1}{2}$
b. $\frac{1}{50}$
c. $\frac{1}{9}$
d. $\frac{2}{9}$
e. $\frac{9}{9}$
f. $\frac{1}{13}$

7. Hey, we just met you, and this is crazy; but here's some numbers, so make them base three.

We missed you so bad. We missed you so, so bad. Also an appropriate Canadian reference! It's always a good time.

a. 9
b. 13
c. 242
d. $\frac{1}{9}$
e. $\frac{1}{13}$
f. $\frac{1}{242}$

8. Write each number as a base-3 "decimal".

a. $\frac{1}{9}$
b. $\frac{1}{2}$
c. $\frac{2}{2}$
d. $\frac{3}{2}$
e. $\frac{1}{13}$
f. $\frac{1}{242}$

Neat Stuff

9. Under what circumstances will a base-10 decimal repeat?

Perhaps it wasn't heard the first time.

10. Under what circumstances will a base-3 decimal repeat?

11. The repeating decimal $.\overline{002}$ means .002002002 . . . . But what number is it? That depends on the *base*! Ace this problem by finding the base-10 fraction equal to $.\overline{002}$ in each given base.

Psst: You did some work on base 3 already. Ace this base problem, and you'll see the sine.

a. base 3
b. base 4
c. base 5
d. base 7
e. base n
f. base 2?!

12. We overheard Amari and Grace debating about whether or not the number .99999 . . . was equal to 1. What do you think? Come up with a convincing argument, and if you already know one, come up with a different one!

We didn't hear who was arguing each side, we mostly just ran away.

13. **a.** Find all positive integers n so that the base-10 decimal expansion of $\frac{1}{n}$ repeats in 3 digits or less.
b. Find all positive integers n so that the base-3 "decimal" expansion $\frac{1}{n}$ repeats in 4 digits or less.

. . . sorry, I'm being told there was a one-character typo in the top note.

14. Patrick has a cube, and he wants to color its faces with no more than two different colors. How many unique colorings are possible? By "unique" we mean that you can't make one look like the other through a re-orientation.

Two of the colorings are possible using PAAS Dip & Dye.

15. Justin has randomized two four-card decks with the same cards: ace, two, three, four. He will flip over the top card of each deck, hoping the same card is in the same position for both decks. If not, he'll flip the next top card from each deck, still hoping.

a. What is the exact probability that Justin will be disappointed?

b. This gives a 1 in N chance of disappointment. Write N to nine decimal places.

16. Write 13 and 242 in base $\sqrt{3}$ instead of base 3. Hee hee hee. Or maybe this turns out to be totally awesome!

Tough Stuff

17. Peg has randomized two thirteen-card decks with the same cards: ace, two, three, four, . . . , jack, queen, king. She will flip over the top card of each deck, hoping the same card is in the same position for both decks. If not, she'll flip the next top card from each deck, still hoping.

This gives a 1 in P chance of disappointment. Write P to nine decimal places.

P is for Peg, and that's good enough for me.

18. Lauren has a cube, and she wants to color its *edges* with two different colors. What now!

19. **a.** Convert 13 to base $\frac{3}{2}$.

b. Convert 13 to base π.

Day 2: Slhi'unfg

"Slhiunfg" is the German word for "shuffling", and also once appeared in an HP Lovecraft story.

Opener

1. Can perfect shuffles restore a deck with 9 cards to its original state? If so, how many perfect shuffles does it take? If not, why not?

Split the cards 5-and-4, and keep the top card on top.

Important Stuff

2. Working with your table, fill in a whole lot of *this* table:

`http://www.tinyurl.com/perfectshuffle`

The file is *in* the computer! Oh, only one computer per table, please.

3. Find the units digit of each annoying calculation. Put those calculators away!

The *units digit* of 90210 is 0, matching Brenda's IQ. We'll miss you, Dylan.

a. $2314 \cdot 426 + 573 \cdot 234$
b. $(46+1)(46+2)(46+3)(46+4)(46+5)$
c. $71^4 \cdot 73^4 \cdot 77^4 \cdot 79^4$

4. Find all possible values for the units digit of each person's positive integer.

You might not want to write a complete list . . .

a. Andrew: "When you add 5 to my number, it ends in a 2."
b. Bethanne: "When you multiply my number by 3, it ends in a 7."
c. Carol: "When you multiply my number by 6, it ends in a 4."
d. David: "When you multiply my number by 5, it ends in a 3. Yup."

5. Unlike "base 10", in *mod 10* the only numbers are the remainders when you divide by 10. In mod 10, 6 + 5 = 1 because 1 is the remainder when 6 + 5 is divided by 10.

Answer all these questions in mod 10.

This is sometimes called *modular arithmetic*. Clock arithmetic is mod 12. Four hours from now, it will be close to 1 o'clock. Apparently time is Irish.

a. $2 + 2 = \boxed{}$
b. $3 \cdot 4 = \boxed{}$
c. $\boxed{} + 5 = 2$
d. $4 \cdot \boxed{} = 2$
e. $5 \cdot \boxed{} = 3$
f. $\boxed{}^4 = 1$

That last one says box to the fourth power, by the way.

6. Repeat the previous problem, except this time do the arithmetic in *mod 7* instead of mod 10.

Good news: there are only 7 numbers in mod 7. Bad news: in mod 7, every Monday is the same.

7. Here's a deck of 18 cards in its starting configuration.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18

Perform perfect shuffles and write down where the cards are located after each shuffle.

a. What is the path taken by card 1?

b. . . . by card 2?

c. . . . by card 3?

Poor card 1, always waiting for the deck to be cut or a mistaken shuffle.

8. Here's a deck of 18 cards in its starting configuration.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

Perform perfect shuffles and write down where the cards are located after each shuffle.

a. What is the path taken by card 0?

b. . . . by card 1?

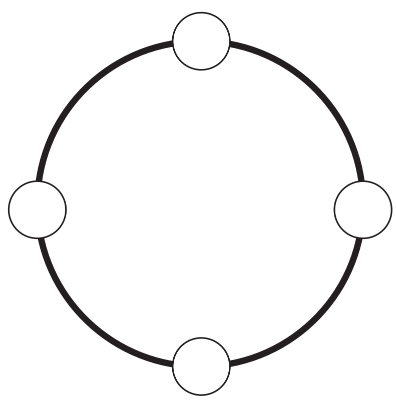
c. . . . by card 2?

Poor card 17, watching all the younger cards mingling.

Neat Stuff

9. Rob is making a pile of bracelets for the students in his class. Each bracelet has four beads on it. Rob has forest green beads and lime green beads.

The school, Green Academy, loves its football team, the Little Green Men. The band mostly plays Green Day, but also some of Al Green's smooth jazz, some family-friendly Cee Lo, and one song by The Muppets. They care a lot about the environment. Favorite movies involve a book, a mile, a lantern, a beret, some tomatoes, some sort of food product non-beloved by Charlton Heston, and Al Gore. They've only got one favorite math theorem, and one favorite dog treat, and one favorite baseball stadium wall. The clocks at this school all run on GMT.



How many unique bracelets can Rob make? If you can turn and/or flip over one bracelet and make it look like another, then we consider them to be the same (non-unique) bracelet design.

10. Go back to the big table that we all filled in together in Problem 2. What cool patterns do you notice?

Some examples of uncool patterns:

"I noticed that most of the values in the table were numbers."

"All the digits in the table could also be found on a computer keyboard."

"Some of the numbers in the table were bigger than others, while others were smaller."

11. We overheard Amari and Grace still yelling about whether or not the number .99999 . . . was equal to 1. Is it? Be convincing.

12. Write each base-10 fraction as a base-3 "decimal". Some of the answers are already given, in which case—awesome!

a. $\dfrac{1}{13} = 0.\overline{002}_3$

b. $\dfrac{2}{13}$

c. $\dfrac{3}{13}$

d. $\dfrac{9}{13}$

e. $\dfrac{10}{13}$

f. $\dfrac{1}{7} = 0.\overline{010212}_3$

g. $\dfrac{3}{7}$

h. $\dfrac{9}{7}$

i. $\dfrac{6}{7}$

Get strategic on these problems: be as lazy as possible. Use tech only if you really have to here.

13. Write the base-10 decimal expansion of

$$\frac{1}{142857}$$

Are there any more interesting ones like this?

14. Repeat Problem 9, but now Rob's bracelets have three beads (equally-spaced) each. What about for five beads? Figure out a general rule for n beads.

15. If $\frac{1}{n}$ terminates in base 10, explain how you could determine the length of the decimal based on n, without doing any long division.

You now know the entire plot of the horrible movie *Terminator 1/4: 0.25 Day*.

16. Sierra wonders what kinds of behavior can happen with the base-10 decimal expansion of $\frac{1}{n}$. Be as specific as possible!

17. Stephen wonders what kinds of behavior can happen with the base-3 decimal expansion of $\frac{1}{n}$.

18. **a.** Suppose $ab = 0$ in mod 10. What does this tell you about a and b?

b. Suppose $cd = 0$ in mod 7. What does this tell you about c and d?

It tells you that a through d hog the spotlight too much. No love for the middle of the alphabet in algebra.

19. Investigate shuffling decks of cards into three piles instead of two. What are the options? Does it "work" like it does with two piles?

20. **a.** Investigate the base-10 decimal expansions of $\frac{n}{41}$ for different choices of n. What happens?

b. Investigate the *base-3* expansions of $\frac{n}{41}$ for different choices of n. What happens?

The fraction $\frac{n}{41}$ is still in base 10 here, so don't convert 41 to some other number.

21. **a.** Find all positive integers n so that the base-10 decimal expansion of $\frac{1}{n}$ repeats in exactly 4 digits.

b. Find all positive integers n so that the base-3 "decimal" expansion of $\frac{1}{n}$ repeats in exactly 5 digits.

22. Write 223 and 15.125 in base 2. Then write them in base $\sqrt{2}$. How cool is that?!

While this problem is cooler than most math, the Supreme Court recently ruled that math cannot actually be cool.

Tough Stuff

23. Faith has an octahedron, and she wants to color its vertices with two different colors. How many unique colorings are possible? By "unique" we mean that you can't make one look like the other through a re-orientation.

24. What about edges? Look for comparisons to Day 1's work on cubes.

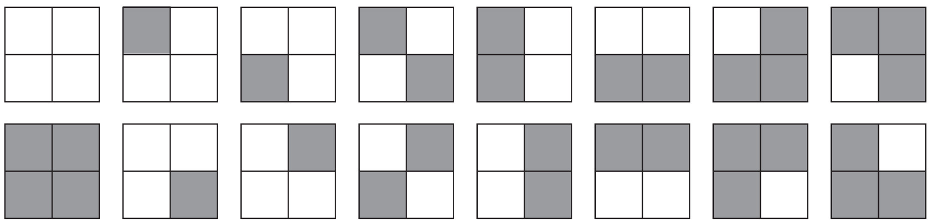
Edges? Edges? We don't need no stinkin' edges!

Day 3: Suln'hfig

Today's title is a rejected name of a Sesame Street character. The character's real name is much harder to spell. Wait, no, it's actually the Klingon word for killing someone while shuffling.

Opener

1. Spencer sells customizable 2×2 chessboards, where each square can be white or black. He says he has to make 2^4 different chessboards because there are two colors to choose from, and four squares that can be independently colored. And here they are:



The other Spencer points out that Spencer doesn't have to make all 16 designs above because he can rotate the chessboard by 90°, 180°, or 270°.

a. What is the smallest number of chessboards that Spencer has to make to be able to get all of the designs above? Or, how many unique colorings of this chessboard exist? Two designs aren't unique if they are rotated versions of each other.

b. Circle the sets of designs above that are considered to be the same via some rotation.

Watch them both on the new Court TV drama *Spencer vs. Spencer*! Both of them are for hire, but they still find time to operate a chain of gift shops.

Important Stuff

2. These are the *entire tables* for addition and multiplication in mod 7. FINISH THEM!!!

+	0	1	2	3	4	5	6
0							
1							
2				5			
3							
4							
5			0				
6							5

×	0	1	2	3	4	5	6
0							
1							
2				6			
3	0						
4							
5			3				
6							1

Today in feeling old: they're now up to Mortal Kombat 11. Fortunately, in mod 7, they're only up to Mortal Kombat 4.

3. What's the difference between *mod 7* and *base 7*? Write a brief explanation (with a numerical example) that a middle-school student could understand.

Bonus points if you can call a middle school student and tell it to them without them hanging up or discussing Fortnite.

4. Use the tables you built to find all solutions to each equation *in mod 7*. Some equations may have more than one solution, while others may have none.

a. $5 + a = 4$
b. $4 \cdot b = 3$
c. $(5 \cdot c) + 6 = 1$
d. $d^2 - 4 = 0$

5. Complete this table, dividing the work among friends!

And foes, I suppose. Hint: Let the foes do 19 and 25.

n	Powers of 2 in mod n	Cycle Length
7	$1, 2, 4, 1, 2, 4, 1, \ldots$	3
9		
11		
13		
15		
17		
19		
21		
23		
25		

6. Describe the path taken by the top moving card when you perfectly shuffle a 14-card deck.

`http://bit.ly/14cards`

Dang, I thought we were done with shuffling. It seems like we shuffle at least once every 24 hours.

7. Describe the path taken by the top moving card when you perfectly shuffle a 22-card deck.

`http://bit.ly/22cards`

8. Today's handout includes the number of shuffles needed for some different deck sizes. Look for some connections to today's work, and see if you can explain them!

Neat Stuff

9. Are there negative numbers in mod 7? Does any number behave like -1?

Sub-zero says hi.

10. Patricia describes another way to shuffle a deck. After creating two equal piles, her first card comes from the pile in her *right* hand instead of her left. For example:

$$123456 \Rightarrow 415263$$

How does this type of shuffling compare to the usual perfect shuffle? What is the same and what is different?

11. Sometimes while shuffling, the deck completely flips. When this happens, all cards appear in reverse order, except for the very top and bottom cards.

a. Find two examples of deck sizes when this occurs.
b. Luke claims that when this happens, that was a halfway point to the shuffling. Is this true? Explain why or why not.
c. Troy claims that this happens at the halfway point of all perfect shuffle sequences. Is this true? Explain why or why not.

Flipping your deck while shuffling sounds like one of the greatest breakdancing moves ever. Breakdancing hasn't been the same since they canceled production on Breakin' 3: The Boogaloo Kid.

12. **a.** Kennya is calculating the powers of 3 in mod 100. Compute the next three entries in the sequence.

$$1, 3, 9, 27, 81, 43, 29 \ldots$$

b. Compute the sequence of powers of 3 in mod 3. Uhh.
c. What about mod 7? What about other mods?

13. **a.** Write $\frac{1}{7}$ as a base-3 "decimal".
b. Write $\frac{1}{100}$ as a base-3 "decimal".

This is not $\frac{1}{9}$. If you'd rather, you can think of it as $\frac{1}{10201}$. Hey, 100 is a palindrome in base 3, nifty.

14. Alissa, Benjamin, and Caleb were standing in the lunch line and had an idea. If any two neighbors switch places, it would create a different arrangement... like

$$ABC \Rightarrow ACB$$

They decided to make a big graph of all six ways they could be arranged, and all the connections that could lead from one way to another. Your turn!

Note that only *neighbors* may switch places. Alissa and Caleb can trade places, but not as the first move.

15. Danny gets in the back of the line behind Alissa, Benjamin, and Caleb, and they realize they're going to be stuck making a much larger graph. Good luck!

This graph will contain lots of little copies of the last graph! Neat.

16. If a number can be represented as a repeating decimal in base 10, does it have to be a repeating decimal in every other base? If yes, explain why. If no, are there any particular bases in which it *must* be a repeating decimal?

17. Predict the length of the base-10 repeating decimal expansion of $\frac{1}{107}$, then see if you were right.

Some of these problems are as confusing as the fact that Steph Curry now hosts a mini-golf game show.

Tough Stuff

18. Predict the length of the *base 2* repeating decimal expansion of $\frac{1}{107}$, then see if you were right.

19. What powers of 10 are palindromes in base 3?

20. For even n, the maximum number of perfect shuffles needed to restore a deck with n cards to its original state appears to be $n-2$. Find a rule that tells you when an n-card deck will require $n-2$ perfect shuffles.

21. *It's p-adic number time!* Every 2-adic positive integer looks like it normally does in base 2, except it has an infinite string of zeros to the left. For example

Where he at, where he at . . . p-adic numbers, p-adic numbers, p-adic numbers with a baseball bat!

$$7 = ...000000000000000111.$$

a. Verify that $7+4=11$ using 2-adic arithmetic.

b. What about subtraction? Try $4-3$, and then try $3-4$.

c. Compute the sum

Oh *man* are you going to have to do a lot of borrowing!

$$...111111111111111. + ...0000000000001.$$

What number is represented by ...11111111111111.?

d. Compute this sum using 2-adic arithmetic:

$$1+2+4+8+16+\cdots=?$$

22. **a.** Find all solutions to $x^2-6x+8=0$ in mod 105 without use of any technology. There's a lot of them.

b. Find all solutions to $x^2-6x+8=0$ in mod 1155 without use of any technology.

Tell you what, we'll give you two solutions: $x=2$ and $x=4$. See, now it's not nearly as tough.

Table for Problem Set 3

This table gives the number of perfect shuffles necessary to restore an n-card deck to its original state.

# cards	# shuffles	# cards	# shuffles
4	2	36	12
6	4	38	36
8	3	40	12
10	6	42	20
12	10	44	14
14	12	46	12
16	4	48	23
18	8	50	21
20	18	52	8
22	6	54	52
24	11	56	20
26	20	58	18
28	18	60	58
30	28	62	60
32	5	64	6
34	10	66	12

Note: Only even values of n listed above to save space. Any deck of n cards with odd n will require the same number of shuffles as a deck with $n + 1$ cards.

Day 4: Shuf'ling Shuf'ling

Opener

1. Each person at your table should pick one of these cards: 3 of ♠ (spades), 9 of ♠, King of ♠, 5 of ♡, King of ♣ (clubs), Queen of ♢. Use this web page to track the location of their card within a standard deck of cards as they undergo perfect shuffles.

`http://tinyurl.com/pcmi52cards`

Then, compare notes with each other.

If there are fewer than six people in your group, each person can track more than one card.

Hm. Are we really going to be shuffling *every day*? At least we should have appropriate music.

Important Stuff

2. Renu sells a line of customizable products featuring equilateral triangles in which each side of the triangle can be red, blue, or green.

Renu allows customers to repeat colors or not. It's customizable!

- **a.** Renu sells customizable t-shirts featuring these equilateral triangles. How many unique t-shirt designs does she need to keep in stock?
- **b.** Renu sells customizable equilateral triangle stickers. How many unique designs will she need to keep in stock, given that the stickers can be rotated? (Two sticker designs are considered non-unique if they are rotated versions of each other.)
- **c.** Renu sells customizable equilateral triangle metal paperweights. How many unique designs will she need to keep in stock, given that the paperweights can be rotated and flipped over?

3. Again, use `http://tinyurl.com/pcmi52cards` to...

- **a.** ... track the numerical positions of the 2 of ♠.
- **b.** ... track the numerical positions of the 4 of ♠.
- **c.** ... track the numerical positions of the 6 of ♠.

Do not use this website to get up, get down, or to put your hands up to the sound.

4.
- **a.** What are the powers of 2 in mod 51?
- **b.** Multiply the numbers in part a by 3 in mod 51.
- **c.** Multiply the numbers in part a by 5 in mod 51.
- **d.** What do you notice?
- **e.** Multiply those numbers by 17 in mod 51. Huh.

5. Explain why, in a 52-card deck, the top moving card returns to its original position after 8 perfect shuffles.

It's times like you would think a Shufflebot would be useful. Sadly, Shufflebot is only programmed to dance and to apologize.

Neat Stuff

6. Use `http://go.edc.org/21cards` to keep track of the position of each card in a 21-card deck as it undergoes perfect shuffles. Complete this table.

Card No.	Positions	Cycle Length	Card No.	Positions	Cycle Length
0	$0, 0, 0 \dots$	1	11		
1			12		
2			13		
3			14		
4	$4, 8, 16, 11, 1, 2, 4, \dots$	6	15		
5			16	$16, 11, 1, 2, 4, 8, 16, \dots$	6
6			17		
7			18		
8			19		
9			20		
10					

7. All 52 cards from the deck we saw on Monday have a cycle. List all 52 cycles. Maybe there is some way to do it without listing the entire cycle for every single card?

8. **a.** In Problem 6, card numbers 4 and 16 were essentially on the same cycle. How many unique cycles does the 21-card deck have?
b. How many unique cycles does a standard 52-card deck have?

LMFAO definitely had some unique cycles, especially when they ran that failed kids' toy store with Schwarz. Leopard prints and shufflebots? No thank you.

9. There are 21 cycle lengths in Problem 6. Add up the *reciprocals* of all 21 numbers . . . surely, this will be a mess . . . oh. What up with that? Look for an explanation.

And we gonna make you lose your mind . . . we just wanna see ya . . . flip that. Doo doo doo do DOO da doo . . .

10. So $2^8 = 1$ in mod 51. All this actually proves is that the *first* moving card returns to its original position after 8 shuffles. Complete the proof by showing that card n, regardless of n, also returns to its original position after 8 shuffles.

11. Pick some deck sizes that you shuffled. Use this to predict the number of powers of 2 in mod n for various n, then verify your prediction by calculating.

Party math is in the house tonight . . . everybody just have a good time.

12. Our favorite repeater, $\frac{1}{7}$, can be written as a "decimal" in each base between base 2 and base 10. Find each expansion and see if they have anything in common.

Step up fast, and be the first one at your table to do base 7.

13. Investigate any connection between the number of powers of 2 in prime mods and the number of powers of 2 in composite mods. Look for an explanation or proof of what you find.

14. Peter, Quiana, and Rasa are waiting in line, wondering if they can get to any arrangement through these two rules:

- The person in the back of the group may jump to the front: $PQR \Rightarrow RPQ$
- The two people at the front of the group may swap places: $PQR \Rightarrow QPR$

Swaps, swaps, swaps! Swaps swaps swaps!

Can all six possible arrangements be made? Make a graph illustrating the options.

15. Shanel joins the back of the group. Under the same rules, decide whether or not all 24 possible arrangements can be made, and make a graph illustrating the options.

One such arrangement is a tribute to the Roman army, the morning after the invasion of the Roman candles!

16. A perfect 52-card shuffle doesn't have a flipped deck at 4 shuffles; we would have noticed that. But does anything interesting happen after the 4th shuffle? Look carefully and compare the deck after 4 shuffles to the original deck. What's the dilly-o?

The observation here may be easier with two decks of cards; one to shuffle, and one to leave in the original setup.

17. Justin handed us a note that said:

$$10^2 + 11^2 + 110^2 = 111^2.$$

a. Surely the numbers 10, 11, 110, and 111 in the note are in base 2. Check to see if the statement is true in base 2.

b. Hey wait, maybe those numbers are in base 3. Check to see if the statement is true in base 3.

c. Oh, hm, maybe it was in base 4.
d. Sorry, it was actually in base n. What!

All your bases are belong to us.

18. *It's p-adic number time!* In 3-adic numbers, non-negative integers are written in base 3 with leading zeros:

$$16 = ...00000000000121.$$

Aww yeah! The p-adic numbers make as much sense as most LMFAO videos.

a. Try $16 - 9$. Hey, that wasn't so bad!
b. Try $16 - 17$. Oh dear.
c. In 3-adic numbers, what real number has the same value of $1 + 3 + 9 + 27 + 81 + \cdots$?

This problem's got that devilish flow rock and roll no halo.

Tough Stuff

19. A stack of 27 little cubes is built to make a bigger 3x3x3 cube. Then, two of the little cubes are removed. Ignoring any gravity effect, how many unique shapes are possible?

It works according to the conversion 27 cubes = 1 Rubik.

20. The length of the repeating decimal for $\frac{1}{2}$ in base p, where p is prime, is sometimes even and sometimes odd. When? Find a rule and perhaps a proof even?

The first person to find and prove this will receive a champagne shower! Offer expires 7/4/2019.

21. For what primes p is there an even length of the repeating decimal for $\frac{1}{5}$ in base p?

22. For what primes p is there an even length of the repeating decimal for $\frac{1}{10}$ in base p?

23. Julia's favorite number is the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$. Julia loves to do arithmetic in base ϕ. The only problem is that even if the only allowable digits in base ϕ are 0 and 1, not every number has a unique representation. Prove that every number has a unique base-ϕ representation if two consecutive 1s are disallowed.

24. Determine and prove the Pythagorean Theorem for p-adic numbers, or decide that this problem is completely bogus and there is no such thing.

It's mathy and you know it.

Day 5: Super Bowl

Opener

1. Complete this table. Don't worry, none of it is in base 2. Long division is fun! While it's a good idea to split the work among one another, please don't use any technology in this work or you may miss some big ideas.

Fraction	Decimal representation	# of repeating digits
1/3	$0.\overline{333}$	1
1/5	0.2	n/a
1/7	$0.\overline{142857}$	6
1/9		
1/11		
1/13		
1/15		
1/17		
1/19		
1/21		
1/23	$0.\overline{0434782608695652173913}$	22
1/25		
1/27		

We are the PCMI Shufflin' Crew. Shufflin' on down, doin' it for you.

Jokes so bad we know we're good. Blowin' your mind like we knew we would.

You know we're just shufflin' for fun, struttin' our stuff for everyone.

Rock paper scissors for who gets stuck doing $1/19$. Or volunteer, ya masochist.

Important Stuff

2. **a.** When doing long division, how can you tell when a decimal representation is about to terminate?
   b. How can you tell when a decimal representation is about to repeat?

3. Explain why the decimal representation of $\frac{1}{n}$ can't have more than n repeating digits.

Digits come, one by one, 'til you shout, "Enough, I'm done!" Then we'll give you even more while you protest . . .

Is there a better upper bound than n digits?

4. Find all solutions to each equation in mod 10.

a. $0x = 1$
b. $1x = 1$
c. $2x = 1$
d. $3x = 1$
e. $4x = 1$
f. $5x = 1$
g. $6x = 1$
h. $7x = 1$
i. $8x = 1$
j. $9x = 1$

In *mod 10*, the only numbers are 0 through 9. For example, $6 + 5 = 1$ and $6 \cdot 5 = 0$. Refrigerator Perry's number changes to 2, but McMahon gets to keep his 9.

5. **a.** Complete this multiplication table for mod 8.

$\times$	0	1	2	3	4	5	6	7
0								
1								
2				6				
3	0							
4								
5								
6			4					
7								1

In *mod 8*, Fridge would be especially unhappy to see his number changed to 0, since he'd be stuck wearing the same number as the punter with the cowbell and Panama hat.

b. How many 1s do you see in your table above?

Don't include the ones on the sidelines.

6. Numbers can multiply with other numbers to make 1! It happens, but not always. Whenever this happens, both numbers are called *units*.

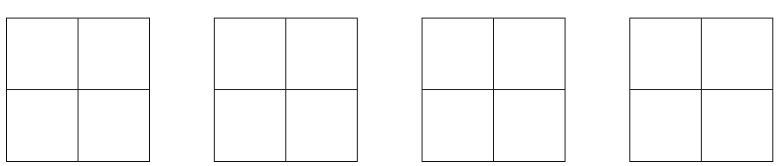
a. If you're working just with integers, what numbers are units?
b. If you're in mod 10, what numbers are units?
c. List all the units in mod 8.
d. List all the units in mod 15.

When working with integers only, 5 is not a unit, since $5 \times \frac{1}{5} = 1$. But . . .

The only time 0 is a unit is in an Algebra 2 book.

7. Here are four uncolored copies of Spencer's 2×2 chessboard. Each uncolored chessboard has four rotational symmetries: you can rotate it by $0°, 90°, 180°, 270°$ about the center and it will look identical.

Rotation by $0°$ *really* doesn't do anything to the chessboard, or any other object, so it's called the *identity transformation.*



This 2×2 chessboard also has four reflective symmetries. Describe the four reflective symmetries of the chessboard by drawing their lines of symmetry above.

8. Joshua and Kim each bought one of Spencer's 2×2 chessboards and painted each of the four squares with their favorite color(s).

They can choose any colors, perhaps the Bears' black and white, or perhaps the hapless Patriots' red and blue. Or even just one color!

a. Joshua says his chessboard has 180° rotational symmetry (about the center). What must be true about the colors of his four squares?

b. Kim says her chessboard has 90° rotational symmetry. What must be true about the colors of her four squares?

c. How many reflective symmetries could Joshua's colored chessboard have?

Neat Stuff

9. Pick some more mods. Try to determine rules for what numbers are units in mod m, and how many units there are. Keep picking more mods until you have a feel for it.

"A Feel For Units" was narrowly rejected as the title for Chaka Khan's greatest hit.

10. Use `http://go.edc.org/15cards` to keep track of the position of each card in a 15-card deck as it undergoes perfect shuffles. Complete this table.

Card No.	Positions	Cycle Length	Card No.	Positions	Cycle Length
0	$0, 0, 0 \ldots$	1	8		
1			9		
2			10		
3	$3, 6, 12, 9, 3, \ldots$	4	11		
4			12	$12, 9, 3, 6, 12, \ldots$	4
5			13		
6			14		
7					

11. **a.** In Problem 10, card numbers 3 and 12 were essentially on the same cycle. How many unique cycles does the 15-card deck have?

b. There are 15 cycle lengths in Problem 10. Add up the *reciprocals* of all 15 numbers. What up with that? Look for an explanation.

I say, oooo weee, what up with that, what up with that?

12. What size decks will get restored to their original order after exactly 10 perfect shuffles (and not in any fewer number of shuffles)?

13. **a.** How many units are there in mod 9? Call this number "Liz".

b. Build a multiplication table for mod 9 *but only include the units*. This multiplication table's size will be Liz-by-Liz.

c. Use `http://go.edc.org/9cards` to track the positions of all cards in a 9-card deck as they undergo perfect shuffles. Hmmm?

14. How many unique cycles did you observe when tracking the locations of cards in the 9-card deck?

15. Can a power of 2 be a multiple of 13? Explain.

16. **a.** Multiply out these expressions and explain what they tell you about $2^n - 1$ when n has factors.

$$(2^a - 1)(1 + 2^a + 2^{2a} + 2^{3a} + \cdots + 2^{(b-1)a}) = ?$$
$$(2^b - 1)(1 + 2^b + 2^{2b} + 2^{3b} + \cdots + 2^{(a-1)b}) = ?$$

b. Find the three prime factors of $2^{14} - 1$ astoundingly quickly, by hand.

17. What size decks will get restored to their original order after exactly 14 perfect shuffles (and not in any fewer number of shuffles)?

18. Determine all deck sizes that can be restored to their original order in 15 or fewer perfect shuffles, and the specific number of shuffles needed for each.

19. Abby, Rob, Suzanne, and Trisha are waiting in line, wondering if they can get to any of their 24 possible arrangements through these two rules:

- The person in the back of the group may jump to the front: TARS ⇒ STAR
- The person in the third position may switch with the person in the first position: TARS ⇒ RATS

Can all 24 possible arrangements be made? Make a graph illustrating the options.

I'm Samurai Mike I stop'em cold. Part of the defense, big and bold.

I've been jammin' for quite a while, doin' what's right and settin' the style.

Give me a chance, I'll rock you good, nobody messin' in my neighborhood.

(This man went on to coach the 49ers.)

Fun Fact: Da Bears were *nominated for a Grammy award* for their performance, and they remain the only professional sports team with a Top 41 hit single. (Wait, there's a Top 41 now?)

I'm mama's boy Otis, one of a kind. The ladies all love me for my body and my mind.

I'm slick on the floor as I can be, but ain't no sucker gonna get past me.

Oh, RATS!

Who decides this? Perhaps it is up to the TSAR. Afterwards, they will head to the museum of fine ARTS and then get some well-deserved RAST. What, that's SART of a word.

20. *It's p-adic number time!* Here are two interesting 10-adic numbers, and we're only going to show you their last six digits. (There are more digits to the left, feel free to try and figure out what they are.)

$$x = ...109376.$$

$$y = ...890625.$$

a. Calculate $x + y$.
b. Calculate the product xy.
c. Calculate x^2 and y^2.
d. How *crazy* are the p-adic numbers?!

Hey I'm Bowen, from EDC. I write books called CME.

I stay up late most every night, writin' problems that come out right.

I add numbers fast, just like magic, but my hairline has gotten tragic.

You all got here on the double, so let's all do the PCMI shuffle . . .

Willie Nelson? Patsy Cline? Aerosmith? Britney Spears? Queen Bey? Gnarls Barkley? Eddie? Horse? Frog? Queen? Train? Madonna is crazy for $U(n)$.

21. Find a mathematical equation that is true in mod 2 and mod 3, but not true in general.

22. Find a mathematical equation that is true in mod 2, mod 3, mod 4, and mod 5, but not true in general.

Psst: you can skip mod 2. Why?

Tough Stuff

23. That thing this morning. How'd we do that?

How'd we do what? You know what. The thing, with the thing.

24. Let n be an integer. Let $U(n)$ be the set of all deck sizes that are restored to its original order after exactly, and no fewer than, n perfect shuffles. (You can disregard trivial decks with 0 or 1 cards.) In Problem 12, you calculated $U(10)$.

a. Prove that $U(n)$ is never empty: there is always some deck for which n shuffles is the lowest possible number.
b. For which n does $U(n)$ contain only one element?

25. **a.** For $n = 1$ through $n = 7$, find all n-digit numbers who last n digits match the original number. For example, $25^2 = 625$, ending in 25.
b. Find a connection between this and the p-adic numbers, or decide that there is no such connection.

My name's Darryl, I'm from LA. I work with math most ev'ry day.

I love to laugh, I love to eat, my Mathematica programs can't be beat.

I've taught kids of every age, now I'm in Park City writin' page by page.

I'm not here to fuss or fumble, I'm just here to do the PCMI Shuffle . . .

Day 6: Cupid

Opener

1. **a.** Find the repeating decimal for $\frac{1}{41}$. List all the remainders you encounter during the long division, starting with 1 and 10.

   b. Write $\frac{100}{41}$ as a mixed number.

   c. Find the repeating decimal for $\frac{18}{41}$. List all the remainders you encounter, including 1 and 10.

   d. Find the repeating decimal for $\frac{1}{37}$. List them remainders!

   e. Find the repeating decimal for $\frac{1}{27}$. Coolio.

No, it's not Coolio, the guy's name is Cupid. He even named the dance after himself, which seems a little presumptuous.

2. Complete this table. Splitting up the work is a great idea, but please do it without fancy spreadsheets or computer programs.

You're going to have to walk it by yourself, now walk it by yourself.

n	Powers of 10 in mod n	Cycle Length
41		
37		
3		
7		
9		
11		
13	$1, 10, 9, 12, 3, 4, 1, \ldots$	6
17		
19		
21		
23	$1, 10, 8, 11, 18, 19, 6, 14, 2, 20, 16,$ $22, 13, 15, 12, 5, 4, 17, 9, 21, 3, 7, 1, \ldots$	22
27		

→ Reminder: You don't have to calculate 10^5 to figure out the 4 in this row. Since you already know the previous 3 is equal to 10^4 in mod 13, you can use $3 \times 10 = 30 = 4$ mod 13. It's super helpful! Also, be careful of Cupid's arrows. This arrow points to the right, to the right.

Important Stuff

3. **a.** What are the rotational symmetries of a regular pentagon? Of a regular hexagon?

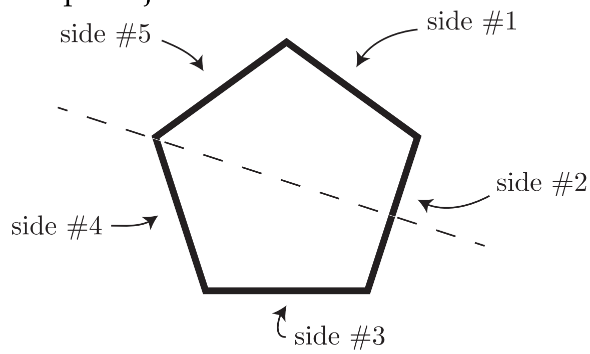
   b. What are the reflective symmetries of a regular pentagon? Of a regular hexagon?

   c. How many rotational symmetries does a regular n-gon have? How many reflective symmetries?

Don't forget the *identity transformation*, which is a rotation by $0°$.

4. **a.** Jeanette bends a paperclip into a regular pentagon. She has 37 colors and paints each of the five sides with some color(s). If her colored paperclip has this reflective symmetry shown below, what must be true about the colors of the five sides? How many possible paint jobs are there?

One option is to paint all five sides in Snugglepuss, and another is to paint four sides Snugglepuss and one side puce.



b. Hannah bends a paperclip into a regular hexagon. She also has 37 colors and paints each of the six sides with some color(s). If her colored paperclip has 120° rotational symmetry, what must be true about the colors of the six sides? How many possible paint jobs are there?

5. **a.** What do these equations have to do with rewriting the number 227 as 1402_5 (in base 5)?

The screen has a color-coded version of this problem. The notorious RGB!

$$\begin{aligned} 227 &= 45 \cdot 5 + 2 \\ 45 &= 9 \cdot 5 + 0 \\ 9 &= 1 \cdot 5 + 4 \\ 1 &= 0 \cdot 5 + 1 \end{aligned}$$

Explain what's happening in each equation above.

b. Here are some equations to help you rewrite 227 in base 3. Fill in the missing numbers.

This must be some sort of spin-off from the original 227. Perhaps an entire show called Jackée about a character not named Jackée.

$$\begin{aligned} 227 &= \quad \cdot 3 + 2 \\ 75 &= 25 \cdot 3 + \\ 25 &= \quad \cdot 3 + 1 \\ &= 2 \cdot 3 + 2 \\ &= 0 \cdot 3 + \end{aligned}$$

c. Rewrite 227 in base 7.

Another 227 spin-off, 22/7, was the first show to prominently feature an approximation of pi. It lasted almost a week.

6. Complete this table. A number x is a *unit* in mod n if there is a number y such that $xy = 1$. In Day 5, many people noticed that this is also the list of numbers in mod n that have no common factors with n.

You can say x is *relatively prime* to n, which is totally different than saying that x is *optimus prime*.

n	Units in mod n	# units in mod n
8	$1, 3, 5, 7$	4
15	$1, 2, 4, 7, 8, 11, 13, 14$	8
25		
30		
49	*too many units*	

← Was that just a *Sneakers* reference? I guess it is now! To the left, to the left.

7. Find the numbers n in the middle column of Problem 2's table that are . . .

a. . . . factors of 9.

b. . . . factors of 99 but not of 9.

c. . . . factors of 999 but not of 9 or 99.

d. . . . factors of 9999 but not of 9, 99, or 999.

e. . . . factors of 99999 but not of 9, 99, 999, or 9999.

f. . . . factors of 999999 but not of . . . alright already.

What's up with that?

Ferris has been absent 9 times. 9 times? NINE TIMES.

I got 99 factored, and 7 ain't one.

Appropriate Beatles song: Number Nine.

Appropriate Nine Inch Nails song: 999999.

Ooooo weeeee, what up with that, what up with that!

8. Calculate each of the following. You may also want to look back at Day 5.

a. $999999 \div 7$
b. $999999 \div 13$
c. $99999 \div 41$
d. $999999 \div 37$

⇒ Only five 9s this time! Cupid's arrow still points to the right, to the right.

Neat Stuff

9. How many units are there in mod 105? Counting them all would be a little painful.

10. Collin points out that the list of units in mod 15 contains 2 and 7, and $2 \times 7 = 14$ is also a unit.

a. Solve $2x = 1$ and $7y = 1$ in mod 15.
b. For x and y above, what is $14xy$ in mod 15?
c. Explain why, if x and y are units in mod 15, then xy is also a unit.

Bonus option: solve part b before solving part a!

11. Lauren points out that the list of units in mod 15 includes some powers of 2: $2, 4, 8, 1$.

a. Write a complete list of all the powers of 2 in mod 15. OK!

b. Explain why, if x is a unit in mod 15, then x^2 is also a unit.

c. Same for x^p for any positive integer power p.

d. Why won't there just be billions of units if you can take any unit to *any* power p and make another one?

OK, Cupid? OKCupid's website says "We use math to get you dates"; its founder has a math degree. The founder probably knows that these powers of 2 can open Jacuzzi doors.

12. What size decks will get restored to their original order after exactly 10 perfect shuffles (and not in any fewer number of shuffles)?

13. For what n does the decimal expansion of $\frac{1}{n}$ have an immediate repeating cycle of 10 digits and no fewer?

For example, $\frac{1}{15} = 0.0\overline{6}$ does *not* immediately repeat.

14. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ be an ordered list in mod 9.

a. Find a number M such that $2M = 1$ in mod 9.

b. Calculate $M \cdot S$ in mod 9. *Keep everything in its original order.*

c. Calculate M^2S in mod 9.

d. Calculate M^kS in mod 9 for all k until something magical happens, then look back to see just how magical it is.

$\Leftarrow$ M^2S can also be used to send pictures to someone's phone. For large k, M^kS indicates that several people enjoyed a meal. To the left, to the left.

15. **a.** What are the units in mod 11?

b. Which of the units in mod 11 are perfect squares? Which are not perfect squares?

c. What are the units in mod 13?

d. Which of the units in mod 13 are perfect squares? Which are not perfect squares?

In mod 11, 5 is a perfect square because $4 \cdot 4 = 5$. It's not the same in mod 13!

16. Euler conjectured that it takes at least k kth powers to add up to another one. For example, $3^2 + 4^2 = 5^2$ but you need three cubes to add up to another cube. In the 1960s this was finally disproven:

$$133^5 + 110^5 + 84^5 + 27^5 = n^5$$

Without a calculator, and hopefully without multiplying it all out, find the value of n.

You fool, Euler! You fool!

17. **a.** There are 16 units in mod 17. Of them, make a list of the eight that are perfect squares and the eight that are not.

b. If x and y are perfect squares, is xy a perfect square... always? sometimes? never?

c. If x is a perfect square and y isn't, is xy a perfect square... always? sometimes? never?

d. If neither x nor y is a perfect square, is xy a perfect square... always? sometimes? never?

I get stupid. I shoot an arrow like Cupid. I'll use a word that don't mean nothin', like looptid. Hey, that guy named the dance after himself, too!

18. There are values of k for which there is only one deck size that restores in k perfect shuffles and no fewer. Are there values of k for which there is only one denominator n such that $\frac{1}{n}$ has a k-digit repeating decimal?

19. A *repunit* is a number made up of all ones: 11111 is a repunit. Investigate the prime factorization of repunits, and determine the values of k for which the k-digit repunit is prime.

20. Hey, what's up with skipping $n = 15$ in Problem 2?

a. What is the length of the repeating portion of the decimal representation of $\frac{1}{15}$?

b. What is the smallest positive integer n for which $10^n = 1$ in mod 15?

c. Aren't your answers for the previous two problems supposed to match? Figure out what's going on and rectify the situation.

Tough Stuff

21. **a.** Expand $(x-1)(x-2)(x-3)\cdots(x-6)$ in mod 7.

b. Calculate $1^6, 2^6, 3^6, \ldots, 6^6$ in mod 7.

c. For p prime and $x \neq 0$, prove $x^{p-1} = 1$ in mod p.

Turns out lots of people get to name dances after themselves, including Dougie, Hammer, Urkel, Freddie, Macarena, Batman, Bartman, those little doll kids, Pee-Wee, Ben Richards, Bernie from Weekend at Bernie's, Chubby Checker, apparently Ketchup, and, of course, Carlton.

22. You can move the top card in a 52-card deck to any position, with a lot less than 52 shuffles. You just need a sequence of both perfect shuffles and Patricia-style shuffles where the right-side card is placed first (see Problem 10 from Day 3). Find a method for moving the top card to any desired position in the deck with six or fewer shuffles.

. . . sorry, I'm being told there was a one-character typo in the top note.

Day 7: iPod

Opener

1. Use the handout from Day 3 to help you fill out this table row-by-row.

n	$2^n - 1$	Factors of $2^n - 1$ (less than 100)	Deck sizes requiring exactly n perfect shuffles (less than 100)
1	1	1	1, 2
2			
3	7	1, 7	7, 8
4			
5			
6			
7			
8	255	1, 3, 5, 15, 17, 51, 85	17, 18, 51, 52, 85, 86
9			
10	1023	1, 3, 11, 31, 33, 93	11, 12, 33, 34, 93, 94
11			

Unlike other tables, this one really needs to be done row-by-row, by everyone. Don't split this work among your tablemates, but do check with one another!

Important Stuff

2. Working with your table, fill in a whole lot of *this* table:

`http://www.tinyurl.com/numberofunits`

A number x is a *unit* in mod n if there is a number y such that $xy = 1$. This is also the list of numbers in mod n that have no common factors with n.

Any table caught violating the instructions in the spreadsheet will be forced to listen to the Super Bowl Shuffle on infinite repeat.

3. What do these equations have to do with $\frac{1}{13}$?

$$\begin{aligned} 1 &= 0 \cdot 13 + 1 \\ 10 &= 0 \cdot 13 + 10 \\ 100 &= 7 \cdot 13 + 9 \\ 90 &= 6 \cdot 13 + 12 \\ 120 &= 9 \cdot 13 + \\ &= 2 \cdot 13 + \\ &= \quad \cdot 13 + \end{aligned}$$

Finish the equations, then use the same method to find the decimal expansions of $\frac{2}{13}$ and $\frac{1}{77}$.

Oops, we didn't finish it all! For a colorful version, see the online class notes. If you're already looking at the online class notes, please stop reading this sentence... now. (Good.)

Free Slurpee Day is coming . . . very soon.

4. **a.** List the powers of 10 in mod 77.
b. How long is the repeating decimal for $\frac{1}{77}$?
c. What is $999999 \div 77$?
d. Explain why the length of the repeating decimal for $\frac{1}{n}$ is the same as the length of the cycle of powers of 10 in mod n.

I'm at the Pizza Hut, I'm at the Taco Bell. I'm at the Taco Bell, I'm at the Pizza Hut. I'm at the permutation Pizza Hut and Taco Bell!

5. Complete this table with help from techmology. Wooo!

n	# units in mod n	$2^{(\text{\# units in mod } n)}$ in mod n	$10^{(\text{\# units in mod } n)}$ in mod n
13			
17			
21			
41			
51			
77			

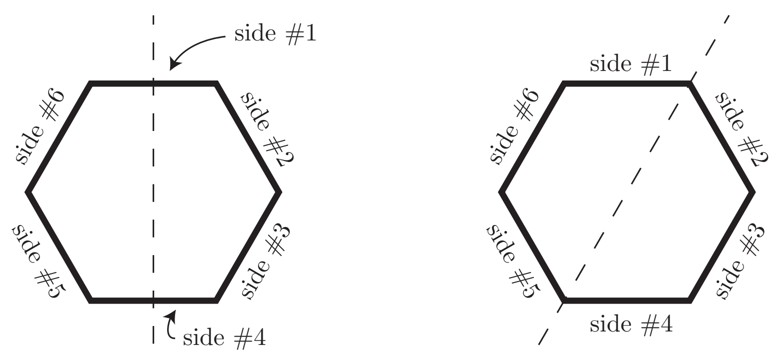
We recommend using `wolframalpha.com` instead of your calculators. For example, try typing `10^6 mod 7` into the box.

What implications does this table have for shuffling cards and repeating decimals?

Yo. Science, what is it all about. Techmology. What is that all about? Is it good, or is it wack?

6. Maria and Marissa both bend a paperclip into a regular hexagon. Both of them paint each of the six sides of their paperclip from a palette of 227 colors.

Repeated colors are allowed. Next year's course will mostly be about the art of paper clip folding, which is usually called borigami.



a. Maria wants her paperclip to have the reflective symmetry shown on the left above. How many possible ways can her paperclip be painted?
b. Marissa wants her paperclip to have the reflective symmetry shown on the right above. How many possible ways can her paperclip be painted?

Yo. Writing out a number like 2,655,237,841 is probably not necessary.

Neat Stuff

7. Suppose you know that in mod n, there's an integer $x \neq 1$ that makes $x^{10} = 1$.

a. Find some other integers k for which you are *completely sure* that $x^k = 1$.

b. Find some integers $k < 10$ for which it is *possible* that $x^k = 1$.

c. Find some integers $k < 10$ for which it is *definitely impossible* that $x^k = 1$.

8. **a.** The decimal expansion of $\frac{1}{7}$ is $.\overline{142857}$. Use this to find the decimal expansions of all $\frac{n}{7}$ for $0 \leqslant n \leqslant 7$.

b. The decimal expansion of $\frac{1}{13}$ is found in Problem 3. Use this to find the decimal expansions of all $\frac{n}{13}$ for $0 \leqslant n \leqslant 13$.

c. What is the same and what is different about the set of expansions of $\frac{n}{7}$ and $\frac{n}{13}$?

9. Edward performs perfect shuffles on a 12-card deck. Write out the order of his cards after a few shuffles.

0	1	2	3	4	5	6	7	8	9	10	11

After Edward completes the shuffles, he sparkles! Also, he hates werewolves, and Jacob, and has a team.

10. Let $H = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ in mod 11.

a. Find a number A such that $2A = 1$ in mod 11.

b. Calculate $A \cdot H$ in mod 11. *Keep everything in its original order.*

c. Calculate A^2H in mod 11.

d. Calculate A^kH in mod 11 for all k until something interesting happens. You decide what counts as interesting!

A^2H indicates that you're at the dentist after too many snacks. For large k, A^kH indicates that zombies are nearby.

11. Lila suggests you look for connections between Problems 9 and 10.

12. Use `http://go.edc.org/27cards` to figure out how many unique cycles are made when a 27-card deck undergoes perfect shuffles. (Refer back to Problem 10 on Day 5.) Can you find a more efficient way to count the number of unique cycles?

Sheesh, nobody told us that *every day* we'd be shuffling. Oh, they did? Still, it seemed like such an empty threat.

13. **a.** There are 6 units in mod 7. For each number x in mod 7, compute x^6 in mod 7. Cool!

Cool. Cool cool cool.

b. There are 4 units in mod 8. For each number x in mod 8, compute x^4 in mod 8. Cool?

NOT COOL!!

14. Suppose you know that in mod n, $ab = 1$ and $cd = 1$.

a. Explain why a, b, c, and d are all units in mod n.

b. Find something that solves this equation:

$$(ac) \cdot (\qquad) = 1$$

This proves that the product of two units is a unit.

c. Find something that solves this equation:

$$(a^k) \cdot (\qquad) = 1$$

This proves that the power of a unit is a unit, and the daughter of a Zappa is also a unit.

15. **a.** Write out all the units in mod 15.

b. Pick one of them and call it v. Now multiply v by all the units in mod 15 (including itself). What numbers do you get?

c. If the units are called $u_1, u_2, u_3, \ldots, u_8$ show that

$$u_1 \cdot u_2 \cdot u_3 \cdots u_8 = (v \cdot u_1) \cdot (v \cdot u_2) \cdot (v \cdot u_3) \cdots (v \cdot u_8)$$

d. Show that $v^8 = 1$ in mod 15.

16. The decimal expansion of $\frac{1}{7}$ is $0.\overline{142857}$. Now, split the repeating digits in half and add them together:

$$142 + 857 = 999.$$

Try this with other fractions $\frac{1}{n}$ that have an even number of repeating digits. Any ideas why this works? What about some other splits?

Some examples include $\frac{1}{13}$, $\frac{1}{73}$, $\frac{1}{91}$, and for the adventurous, $\frac{1}{17}$ and $\frac{1}{23}$.

17. One way some people counted up the units in mod 105 was this: *I'll take all 105 numbers, then subtract the number of multiples of 3, then 5, then 7. Oh, wait, then I have to add back in the multiples of 15, 21, and 35. Oh, wait, then I have to subtract the multiples of 105.*

a. Does this work? There are 48 units in mod 105.

b. What's the probability that a number picked between 1 and 105 *isn't* a multiple of 3? 5? 7?

c. Consider the product $(3 - 1)(5 - 1)(7 - 1)$.

Yes, and . . . ??

18. Change the argument a little from Problem 15 to show that for any unit in any mod,

$$u^{(\text{\# units in mod } n)} = 1 \text{ in mod } n$$

19. Nathan watches 6♠ (spades) as a 52-card deck undergoes perfect shuffles. Before each shuffle, he notes whether the 6♠ is in the left half (aka the top half) or the right half of the deck. Write a 0 if the card is in the left half, and a 1 if it is in the right half. Cupid says you should do it again with 10♢, and there is room for you to pick your own card to try.

See the animations at `tinyurl.com/pcmi52cards`. I heard the ace of spades is cosmic.

To the right, to the right . . . come on, you know the Cupid Shuffle! And so does 10♢, apparently!

Card	Which half just before shuffle #?							
	1	2	3	4	5	6	7	8
6♠								
10♢								

a. For each card, compute the base 2 number that corresponds to the eight 0s and 1s you listed.

These numbers run from 0 to 255.

b. Given a card, is there a way to construct its sequences without watching the shuffles?

Since you asked, probably, yes?

20. Abraham hands you his favorite multiple of 5, written as an eight-digit base 2 number: $011010??_2$. Oh noes, you can't make out the last two digits. What must those missing digits be for the number to be a multiple of 5?

Tough Stuff

21. Consider different odd primes p and q. The number p may or may not be a perfect square in mod q, and the number q may or may not be a perfect square in mod p. Seek and find a relationship between these two things! Respek.

There is so little respek left in the world, that if you look the word up in the dictionary, you'll find that it has been taken out.

22. Consider different odd primes p and q. The length of the repeating decimal of $\frac{1}{p}$ in base q may be odd or even, and the length of the repeating decimal of $\frac{1}{q}$ in base p may be odd or even. Seek and find a relationship between these two things! Booyakasha.

Day 8: Slurpee Mix

Opener

1. What are the powers of 10 in mod 21, and how many are there?

2. Find each decimal expansion in base 10. Try to complete the table with the least amount of computation! For example, Christian suggests that you fill in 10/21 first.

If you use a calculator, report your answers as repeating decimals instead of rounding them off.

Fraction	Decimal	Fraction	Decimal
1/21	$0.\overline{047619}$	11/21	
2/21		12/21	
3/21		13/21	
4/21		14/21	
5/21		15/21	
6/21		16/21	
7/21		17/21	
8/21		18/21	
9/21		19/21	
10/21		20/21	

Hey, me just met you, and this is crazy, but you got Slurpee, so share it maybe?

Important Stuff

3. **a.** How many fractions in the table above are in lowest terms?
b. How many units are there in mod 21?
c. Pick three different units in mod 21. For each, calculate u^{12} in mod 21.

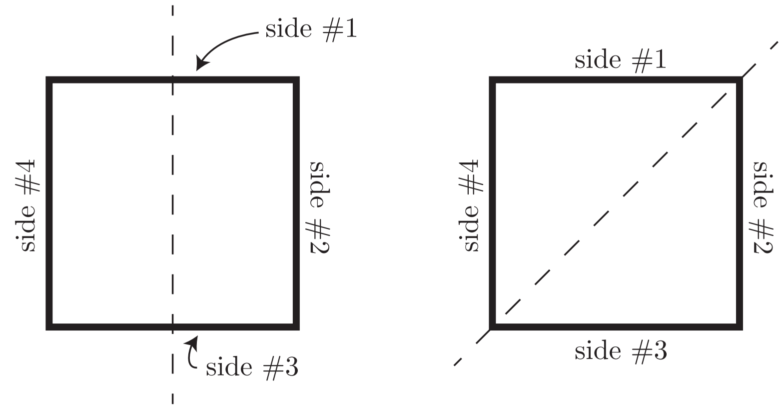
7/11 is in lowest terms, but the ripoff copy store 6/12 is not.

4. Describe any patterns you notice in the table above. What fractions form "cycles" that use the same numbers in the same order? Write out each cycle in order. How many unique cycles are there? How long are the cycles?

Coicles? I don't see any coicles here, nyuk nyuk nyuk.

5. Gabe and Gary both bend a paperclip into a square. Both of them paint each of the four sides of their paperclip from a palette of k colors.

While you're welcome to call these Square One and Square Two, only one of them can host Mathnet.



a. Gabe wants his paperclip to have the reflective symmetry shown on the left above. How many possible ways can his paperclip be painted?

b. Gary wants his paperclip to have the reflective symmetry shown on the right above. How many possible ways can his paperclip be painted?

Your answers here will be expressions with k in them. This problem has squares and k because Circle K doesn't sell Slurpees.

6. How do these equations relate to the base 2 "decimal" for $\frac{1}{21}$?

$$\begin{aligned}
1 &= 0 \cdot 21 + 1 \\
2 &= 0 \cdot 21 + 2 \\
4 &= 0 \cdot 21 + \\
8 &= \quad \cdot 21 + 8 \\
16 &= 0 \cdot 21 + \\
32 &= 1 \cdot 21 + \\
22 &= 1 \cdot 21 +
\end{aligned}$$

As per use, we have a nice version in fabulous color on screen. While it is possible to get a Slurpee with red, blue, and green flavors, we don't recommend drinking it.

Finish the equations above.

7. What are the powers of 2 in mod 21, and how many are there?

8. Let's try the opener again, but this time we'll use *base 2*. Find each "decimal" expansion. Try to complete the table with the least amount of computation. For example, Monika suggests that you fill in 2/21 first.

If you use a calculator, good luck finding a calculator that reports answers in base 2! You'd be better off using a deck of cards.

Fraction	Base 2 "Decimal"	Fraction	Base 2 "Decimal"
1/21	$0.\overline{000011}$	11/21	
2/21		12/21	
3/21		13/21	
4/21		14/21	
5/21		15/21	
6/21		16/21	
7/21		17/21	
8/21		18/21	
9/21		19/21	
10/21	$0.\overline{011110}$	20/21	

From Cookie Monster's "Share It Maybe": Me got a wish on my mind, it is a chocolate chip kind. Me look at you and me tell, you might have snickerdoodle. Me start to really freak out, please someone call a Girl Scout. Don't mean to grumble or grouse, this taking toll on my house. Me going off my rocker, please give me Betty Crocker.

9. Here is a 21-card deck being perfectly shuffled.

`http://go.edc.org/21cards`

Follow some cards and follow some remainders. How can you use the cards to find the *entire base 2 expansion* for each fraction?

Wait, . . . we *are* better off using a deck of cards? Ohh my.

Neat Stuff

10. **a.** Write out all the units in mod 21.

b. Write out all the powers of 10 in mod 21 (again), starting with 1 and 10.

c. Take all the powers of 10 in mod 21 and multiply them by 2. What happens?

d. Take all the powers of 10 in mod 21 and multiply them by 3. What happens?

e. Take all the powers of 10 in mod 21 and multiply them by 7. What happens?

Hey, you're still in mod 21! There's no such number as 32.

11. Complete this table.

n	"Decimal" for $1/n$ in base 2	# of repeating digits	# of units in mod n
3	$0.\overline{01}$	2	2
5			
7			
9			
11			
13			
15			
17			
19	$0.\overline{000011010111100101}$	18	
21	$0.\overline{000011}$	6	

Using the method from Problem 6 may be helpful to your sanity, but you can also use long division. Split the work, but keep away from techmology or you may miss some big ideas.

12. Cut out the 2×2 chessboards, one for each person in your group. The uncolored chessboard has four rotational symmetries (0°, 90°, 180°, and 270° counterclockwise) and four reflective symmetries (—, ╲, |, and ╱).

a. Maria tells you to rotate your chessboard by 90° counterclockwise, then to reflect it (flip it over) along the — (horizontal) axis. Those two operations are equivalent to a single reflection. Which reflection is it?

b. Marila tells you to reflect your chessboard along the — (horizontal) axis, then rotate it by 90° counterclockwise. Those two operations are equivalent to a single reflection. Which reflection is it?

Though the chessboard is uncolored, they are labeled so that you can keep track of which square is which. The four squares are labeled like the four quadrants of the coordinate plane.

These moves were popularized by Daft Punk in their hit song "Turn It Flip It, Flip It Turn It". This is not an invitation to turn the square one more time.

13. To fill in each space in the table on the next page, first take your 2×2 chessboard and perform the transformation matching the column of the space, then do the transformation matching the row. Fill in the space with the transformation that is equivalent to the composition of those two transformations.

·	Rot 0°	Rot 90°	Rot 180°	Rot 270°	Refl —	Refl ╲	Refl │	Refl ╱
Rot 0°								
Rot 90°		Rot 180°						
Rot 180°						Refl ╱		
Rot 270°								
Refl —								
Refl ╲					Rot 270°			
Refl │								
Refl ╱								

14. Compare and contrast Problem 13 with Problem 5 from Day 5.

I noticed this table was wider. I wonder how an even wider table might look on this page.

15. The table you built in Problem 8 has a pile of repeating base 2 "decimal" parts. But what about the numbers *within* the repeating digits?

a. The base 2 "decimal" for $\frac{10}{21}$ is $0.\overline{011110}$. What base 10 number is made from those repeating digits?

b. Do this a few more times for other n until you can explain how to write the base 2 "decimal" for any $\frac{n}{21}$ directly.

16. OK, so about that magic trick we've done once or twice. How'd we do that? The work in Problem 15 is helpful, but you'll need to make some serious modifications. The work in Problems 19 and 20 on Set 7 is also helpful.

17. **a.** If you haven't yet, go back and do Problem 15 from Set 7.

b. The *order* of a unit u in mod n is the smallest power $k > 0$ such that $u^k = 1$ in mod n. Prove that the order of any unit u in mod n must be a factor of the number of units in mod n.

You've got to go back Marty! Back to . . . oh, the past. Bah.

Anarchy in the u^k!

18. The decimal expansion of $\dfrac{1}{7}$ is $0.\overline{142857}$. Now, split the repeating digits in thirds and add them together:

$$14 + 28 + 57 = 99.$$

Try this with other fractions $\frac{1}{n}$ whose repeating digit lengths are multiples of 3. What's up with that!

Slurpee sales on July 11 are typically about 40% *higher* than normal, even though they are also being given away for free. Sometimes, if your timing is bad, Slurpee sales are also soupy sales.

19. So, shuffling. Madhu is so good at it that she can split a deck of cards into three equally-sized piles and perfect-shuffle them.

You try it too. Pick a number of cards that is a multiple of three and perform her perfect triple shuffles. See what you find. We recommend numbering your cards as 0, 1, 2, ... There's a lot to find!

Just ask her! Okay maybe we are lying. But we told the truth about that Cookie Monster song, maybe.

Curly: I'm tryin' to think, but nothin' happens!

20. Suppose a and b are relatively prime. How does the length of the decimal expansion of $\frac{1}{ab}$ compare to the lengths of the decimal expansions of $\frac{1}{a}$ and $\frac{1}{b}$?

21. In a Reader Reflection in the *Mathematics Teacher* (March 1997), Walt Levisee reports on a nine-year-old student David Cole who conjectured that if the period of the base-10 expansion of $\frac{1}{n}$ is $n-1$, then n is prime. Prove David's conjecture, and discuss whether it might apply to other bases.

Warning: whatever Slurpee flavor you pick, do not pick the "Mystery Flavor". Stores take whatever flavor is not selling at all, and relabel it as "Mystery Flavor" hoping to sell some more of it. So, heed this warning, unless you want your Slurpee to taste like Crunch Berries. (You don't.)

Tough Stuff

22. The Fibonacci numbers start with $F(0) = 0, F(1) = 1, F(2) = 1$, etc:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \ldots$$

Show that if m is a factor of n, then $F(m)$ is a factor of $F(n)$. For example, $F(7) = 13$ is a factor of $F(14) = 377$.

23. Show that if F is a Fibonacci number greater than 2, then neither $F^2 - 1$ nor $F^2 + 1$ is prime.

24.
- **a.** Prove that if p is an odd prime, there is at least one base $b < p$ in which the expansion of $\frac{1}{p}$ has period $p - 1$.
- **b.** Determine the number of such bases in terms of p.

Quadrant II	Quadrant I
Quadrant III	Quadrant IV

Quadrant II	Quadrant I
Quadrant III	Quadrant IV

Quadrant I	Quadrant II
Quadrant IV	Quadrant III

Quadrant I	Quadrant II
Quadrant IV	Quadrant III

Day 9: Shuffles, Assorted

Opener

1. Use this set of equations to find the base 2 "decimal" for $\frac{5}{51}$.

$$
\begin{aligned}
5 &= 0 \cdot 51 + 5 \\
&\ . \\
10 &= 0 \cdot 51 + 10 \\
20 &= 0 \cdot 51 + 20 \\
40 &= 0 \cdot 51 + 40 \\
80 &= 1 \cdot 51 + 29 \\
58 &= 1 \cdot 51 + \\
&= \quad \cdot 51 + \\
&= \quad \cdot 51 + \\
&= \quad \cdot 51 + 5
\end{aligned}
$$

Thank goodness there are no obnoxious giant tables on this problem set. Let's celebrate! As is customary, we gotta get down.

What would William Shatner yell if he was upset with the pedagogical style of a tutoring website? Answer later.

Important Stuff

2. **a.** When you multiply a number by 10 in base 10, what happens to the decimal point?
   b. When you multiply a number by 2 in base 2, what happens to the "decimal point"?
   c. The base 2 "decimal" for 16/51 is $0.\overline{01010000}_2$. What's the base 2 "decimal" for 32/51?
   d. Based on your previous answer, what's the base 2 "decimal" for 13/51?

The *Curly Shuffle* is a 7-cycle on the floor: "Woooooooooo woo woo woo, woo woo woo. Woooooooooo woo woo woo, woo woo woo." See also: Angus Young in Let There Be Rock; Homer Simpson in the "Lisa needs braces" episode.

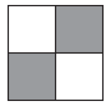
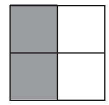
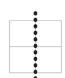
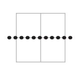
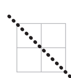
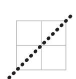
3. **a.** Grahame demands you write all the powers of 2 in mod 51, starting with 1.
   b. Take all the powers of 2 in mod 51 and multiply them by 2. What happens?
   c. Take all the powers of 2 in mod 51 and multiply them by 5. What happens?
   d. Take all the powers of 2 in mod 51 and multiply them by 17. What happens?

Dave and Brian McKnight both agree that the list of powers ends back at 1.

Like Anthrax, you're caught in a mod! Once you're in a pattern you are stuck. Stuck in a coicle, nyuk nyuk nyuk.

4. **a.** Jaime and Jennifer ordered t-shirts with these two logos on the front. Fill in this table for what their t-shirt would look like if their logo was transformed as described.

Hmm, isn't this an obnoxious giant table? **No!** It is fine, just really annoying how it's table-spreading and taking up the whole page. Awww man!

Original	Jaime: 	Jennifer: 
Rot 0° ↺		
Rot 90° ↺		
Rot 180° ↺		
Rot 270° ↺		
Refl 		
Refl 		
Refl 		
Refl 		

Remember that the rotate by 0° transformation is an *identity transformation* that doesn't do anything to the object.

Ali: "Now hold it. The *Ali Shuffle* is a dance that will make you scuffle. During the time that I'm doing this shuffle, for a minute, you're going to be confused. You must get in a boxing position, and have a little dance." Howard Cosell – do the voice: "What we've just seen perhaps is the heavyweight champion of the world in what should be his true profession, that of a professional dancer."

b. A *symmetry* of an object is just a transformation of the object that leaves it looking identical. Of all the possible transformations of the 2×2 chessboard, which symmetries does Jaime's design have? Jennifer's?

c. How many total possible symmetries are there? How many does Jaime's design have? How many does Jennifer's design have?

The *Ickey Shuffle* is the most famous touchdown dance of all time. One of the two TV announcers from the movie *Cars* did the Ickey Shuffle after winning a NASCAR race. For more information, watch the "Return of the Shirt" episode from *How I Met Your Mother*. It's . . . wait for it . . .

5. **a.** Without converting to a fraction, find a base-10 decimal such that

$$0.\overline{291594} + 0.\overline{mariah} = 1$$

It's like Mariah's "Get Your Number".

b. Without converting to a fraction, find a base 2 decimal such that

$$0.\overline{011001} + 0.\overline{ishrat} = 1$$

c. In the obnoxious table, how do the base 2 expansions of $\frac{1}{51}$ and $\frac{50}{51}$ compare?

Wait, there's an obnoxious table? Oh no. *NOOOOOO!* Ask Bowen and Darryl for it.

6. Complete the obnoxious table. Be lazy and avoid the use of techmology.

Oh, bother.

7. Describe any patterns you notice in the obnoxious table. What fractions form "cycles", using the same numbers in the same order? Write out each cycle in order, and find all the cycles. How many unique cycles are there?

8. Here is our happy little 52-card deck doing happy little perfect shuffles, just like the very first time we saw it.

`http://tinyurl.com/pcmi52cards`

Follow some cards. Follow some remainders. Figure out how you can use the cards to find the *entire base 2 expansion* for a fraction in the form $\frac{n}{51}$.

Bob Ross's hair was permed so that he could save money on haircuts. Before his art career, he was a sergeant at an Air Force base in Alaska, where presumably he saw many happy mountains and trees. After leaving the military he vowed never to scream again. Bob Ross went on to become a painter beloved by many, including the creators of "Friends", who named David SchwimmerâĂŹs character after him.

Neat Stuff

9. Let's look at some eight-digit numbers in base 2.

a. What number is 00011001_2 in base 10?

b. One of the fractions in the obnoxious table has a base 2 expansion of $0.\overline{00011001}$. Which one? Try to figure out how you might find that fraction without looking at the obnoxious table, and without staring at cards like crazy.

c. Start over with 00101101_2 and $0.\overline{00101101}$.

d. Try it again with 11110000_2 and $0.\overline{11110000}$.

Cupid loves you! To the right . . .

e. What are the largest and smallest possible values of an eight-digit number in base 2?

f. Can you explain why all the entries in the obnoxious table turn out to be multiples of [REDACTED]?

At least it's not an obnoxious multiple.

10. **a.** The base 2 number $110010ab_2$ is a multiple of 5. What are the missing digits, and what multiple of 5 is it?

. . . legendary!!

b. The base 2 number $001101ab_2$ is a multiple of 5. What are the missing digits, and what multiple of 5 is it?

c. Make your own: pick the first six digits of a base 2 number and try to find the missing digits.

d. How does our magic trick work?

What?? You mean it's not magic? No, and neither was getting Lauren to pick the card that does the Cupid Shuffle. Thank you, Lauren!

11. Based on the card animations, what do you get for the base 2 expansion for $\frac{51}{51}$? Interesting.

12. **a.** Take all the powers of 2 in mod 51 and multiply them by k. That was fun. Whee! It will be helpful, we promise.

Stay. You only hear what you want to.

b. A 52-card deck returns to its original position in 8 perfect shuffles. But some cards return sooner. Find an equation that would be true for any card k that returns to its original position after 2 shuffles, then solve it.

Zero bottles of beer on the wall, zero bottles of beer. You take one down and pass it around, 50 more bottles of beer in mod 51.

c. Find an equation that would be true for any card k that returns to its original position after 3 shuffles, then solve it.

d. FOUR!

That's not even a question. But don't worry, be happy.

13. Use ideas from Problem 12 to show that if u is a unit, then u must be part of a cycle that has the same length as the cycle containing 1.

14. A 36-card deck returns to its original position in 12 perfect shuffles. Determine all the cards that return sooner, and the cycle length of each.

15. Here are some shuffle animations. Triple shuffles!!! We promise they're cool.

From *Hollywood Shuffle*: Ain't nothin' to it, but to do it! This movie featured Keenen Ivory Wayans as "Jheri Curl" (he also wrote the movie).

`http://go.edc.org/3piles`

Go figure stuff out. Decimals in base 3, powers of 3 (in what mod?), cycles, magic tricks, all that jazz. What stays the same? What changes?

16. Determine all cycles of cards in the triple shuffle, and the cycle length of each. Given the number of units in the mod, what cycle lengths are possible?

It continues to be the case that you are shuffling in a typical siderial period.

17. **a.** Find a number x so that $x = 1$ in mod 11 and $x = 0$ in mod 13.

b. Find a number y so that $y = 0$ in mod 11 and $y = 1$ in mod 13.

c. Find a number z so that $z = 5$ in mod 11 and $z = 6$ in mod 13.

KHAAAAAANNNNNNNN!

18. Find a number M so that $M = 2$ in mod 3, $M = 3$ in mod 5, $M = 4$ in mod 7, and $M = 5$ in mod 11. Try this without techmology and with as little guesswork as possible!

Will computahs ever be able to work out what 999999999999999999 999999999999999999 multiplied by 9999999 ... yo I haven't finished you don't know what I was going to say ... 99899999999.889988999 9999999999 is?

19. Look back at the table from Day 8 you made for the fractions from $\frac{1}{21}$ through $\frac{20}{21}$. All the digits from 0 through 9 are used, and there are 120 total digits in the repeating parts of the expansions.

a. Which digit is used most frequently in the decimal part of the expansions? Which digit is used least frequently?

b. Does this happen in other mods? Other bases?? So cool.

Okay, it's still not cool, but cool by math standards. Not cool like Kool Moe Dee cool, but hey.

20. Today is 7/12/19, and $7 + 12 = 19$. Oh snap!

a. How many more times this century will there be a day like this? By *this* we mean the next one is August 11, 2019.

And the last one is . . . later than the others?

b. How many times will Buck Rogers in the 25th Century see a day like this? You may assume that Buck Rogers arrives on January 1, 2401 and remains alive through the entire century.

Wow, that is a really cool answer!

c. How can the second answer help you check the first?

Tough Stuff

21. Figure out a way to shuffle cards so that you could use the cards' positions after each shuffle to read off base-10 decimal expansions of fractions . . . or prove it is impossible to do so.

22. Prove that there are infinitely many primes p for which the decimal representation of $\frac{1}{p}$ has cycle length $p - 1$.

23. Take a grid of circles, 12-by-21, and color each circle with one of four colors. Either find a coloring that does not produce a monochromatic rectangle – a rectangle with all four corners the same color – or prove that such a coloring is impossible.

24. Find a fraction that has the Fibonacci numbers in its decimal expansion:

$$0.001001002003005008013021034\ldots$$

25. Find a fraction has the square numbers in its decimal expansion:

$$0.001004009016025036049064081\ldots$$

Dang, there's an entire half-page here for another obnoxious table! Womp womp.

Table for Problem Set 9

Oh my. This is an obnoxious table.

Fraction	Base 2 "Decimal"
1/51	$0.\overline{00000101}$
2/51	
3/51	
4/51	
5/51	
6/51	
7/51	
8/51	
9/51	
10/51	
11/51	
12/51	
13/51	
14/51	
15/51	
16/51	$0.\overline{01010000}$
17/51	
18/51	
19/51	
20/51	
21/51	
22/51	
23/51	
24/51	
25/51	

Fraction	Base 2 "Decimal"
26/51	
27/51	
28/51	
29/51	
30/51	
31/51	
32/51	
33/51	
34/51	
35/51	
36/51	
37/51	
38/51	
39/51	
40/51	
41/51	
42/51	
43/51	
44/51	
45/51	
46/51	
47/51	
48/51	
49/51	
50/51	$0.\overline{11111010}$

Day 10: The Band

Opener

1. Hey! Let's watch 21 cards under perfect shuffles: `http://go.edc.org/21cards`. Each person at your table will pick one of these six cards:

$$0, 2, 3, 5, 7, \text{ or } 9.$$

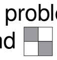
Write down where your card is located after Shuffle #0, Shuffle #1, . . . , Shuffle #5, and compare with your group. The set of positions formed this way is the *orbit of your card under shuffling*, and the number of things in that set is the *size of the orbit*.

We keep track of six shuffles because that is the number of perfect shuffles needed for a 21-card deck.

Orbits? You've seen them previously, but we called them *cycles* or coicles, nyuk nyuk nyuk. They are not circles, nor rotaries.

2. As a group, please review your work for Problem 1 on Day 3 and Problem 2 on Day 4 and reach consensus on your answers.

3. Each person at your table will pick one of Spencer's 16 colorings of the 2×2 chessboard from Day 3. What is the orbit of your coloring under the eight transformations (see Problem 4 on Day 9)? What is the size of your orbit? Compare with those at your table.

The orbit of your coloring is the set of things you get when you apply all of the transformations to that coloring. In this problem, we consider  and  to be different.

Important Stuff

4. **a.** Fill out Handouts #1, #2, then #3.
b. What do you notice about the numbers in rows 1 and 2 on each handout?

Handouts! Oh we've got handouts! We've got sacks and sacks of handouts . . .

5. **a.** On each handout, add up the numbers in row 3. Compare your answers to the ones you agreed to in Problem 2.
b. Write a one-sentence explanation for what you're observing.

If you did them, you might also enjoy looking back at Problem 9 on Day 4, and Problem 11 on Day 5.

6. Eric tells you his deck of cards will return to its original order in 12 perfect shuffles and no less. He doesn't tell you how many cards are in his deck.
a. Why can't there be an orbit of size 5 in his deck of cards as they get perfectly shuffled?
b. What are the possible orbit sizes for any card in his deck?

He also tells you that you're the saddest bunch he's ever met, and that this problem is as mysterious as the dark side of the moooooon.

7. Regular hexagons have six rotational symmetries (including the identity transformation) and six reflective symmetries. Lauren folds a paperclip into an regular hexagon and paints its sides with some colors. She tells you the size of its orbit under these reflections and rotations is 6.

Lauren minored in borigami.

- **a.** Including the identity transformation, how many symmetries must her paperclip have?
- **b.** Give two examples of how her paperclip might be painted, one with rotational symmetry and one with reflective symmetry.

8. Rob's bracelets were such a success that he now wants to start an online made-to-order bracelet company. To promote his company, he wants to brag about the number of possible bracelets that can be made under different conditions.

What? You forgot about Rob's bracelets? See Problem 9 from Day 2.

Remember that if we can turn and/or flip over one bracelet and make it look like another, then we consider them to be the same (non-unique) bracelet design.

- **a.** How many "unique" bracelet designs can be made if each bracelet has 5 equispaced beads and 2 available bead colors to choose from?
- **b.** . . . six equispaced beads?

Neat Stuff

9. . . . seven equispaced beads?

Bees?? Oh, *beads*!

10. In Problem 2 from Day 4, Renu put an equilateral triangle on different products, limiting the allowable transformations of the triangle.

Why haven't any of you bought a triangle T-shirt from Renu's online store? There are 3^3 T-shirt options!

- **a.** What happens to your answers on Handout #3 if only rotations are allowed (no reflections)?
- **b.** What happens to the sum of the numbers in row 3?

11.
- **a.** Patrick has a cube and paints each face either peach or pink. How many unique colorings are possible? By "unique" we mean that you can't make one look like the other through a re-orientation.
- **b.** Megan also has a cube, but has a palette of three colors to choose from for the faces, instead of two. Estimate how many "unique" colorings there are now.

This is also Problem 14 from Day 1. Choosing every face pink or peach is allowed.

This is a significantly more difficult problem, still less challenging than finding the sun on Saturday. Do not spend a lot of time here. Just estimate and move on.

12. Nancy's made-to-order bracelet company offers three bead colors to choose from. Watch out, Rob! How many unique bracelet designs can be made if each bracelet has 4 equispaced beads? Five beads?

13. Repeat Problem 13 from Day 8, but this time for the symmetries of the equilateral triangle. What similarities and differences do you see between the two tables you generated?

How big is this table, anyway?

14. Build an addition table for $\mathbb{Z}_6$. Wait what? Oh, that's just mod 6. Also, build a multiplication table for $\mathbb{Z}_9$.

$\mathbb{Z}$ stands for the integers, because of course it does. Then $\mathbb{Z}_m$ stands for the integers mod m. Some people live their entire lives under the $\mathbb{Z}$.

$(\mathbb{Z}_6, +)$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1					
2	2					
3	3					
4	4					
5	5					

$(\mathbb{Z}_9, \times)$	0	1	2	3	4	5	6	7	8
0		0							
1	0	1	2	3	4	5	6	7	8
2		2							
3		3							
4		4							
5		5							
6		6							
7		7							
8		8							

15. Build a multiplication table for mod 9 that includes only its units. This table's size will be Liz-by-Liz. The set of units of $\mathbb{Z}_n$ is called $\mathbb{U}_n$. Notice anything?

A unit is a lot like the ratio of bananas to bathrooms . . . no!!! A number u is a *unit* if there is a number v that solves $uv = 1$. The multiplication table should help, but we've also found another rule for deciding if a number in a mod was a unit.

16. Hey, three tables you made on this page are the same size. Are any of the three tables "the same" in any meaningful way?

You can use Problem 15 to model $\mathbb{U}_n$! Now there's a joke only teachers could love.

17. **a.** Find all solutions to $x^2 = 1$ in $\mathbb{Z}_{105}$. There are a lot of them, and busting 105 into little pieces may help.

b. Use completing the square (!) to find all solutions to $x^2 + 24 = 10x$ in $\mathbb{Z}_{105}$.

18. If x is an element of $\mathbb{Z}_n$, show that there must *either* be an element y that solves $xy = 1$, *or* an nonzero element z that solves $xz = 0$, but not both.

Just look at the numbers round you, right behind the math-camp door. Such wonderful mods surround you, what more is you lookin' for!

19. What happens if we modify Handout #3 so that it is for the 22-card deck instead of the 21-card deck? What happens to the sum of numbers in row 3? What's going on with that last card?

20. We now know that 52 shuffles will restore a deck of 54 cards using perfect shuffles. This is true because $2^{52} = 1$ in mod 53 is the smallest power of 2 possible. Suppose you wanted to prove this to a friend, but you only have a simple calculator that can't calculate 2^{52} exactly. Find a way to calculate 2^{52} in mod 53 using as few operations as possible and just a simple calculator. What other powers of 2 would you need to calculate to ensure that it takes 52 shuffles to restore the deck and not some smaller number?

We got the spirit, you got to hear it, under the $\mathbb{Z}$.

The two is a shoe, the three is a tree.

The four is a door, the five is a hive.

The six is a chick, the seven's some surfin' guy . . . *yeah*

The eight is a skate, the nine is a sign.

And oh that zero blow!

21. **a.** Without a calculator, determine the number of perfect shuffles it will take to restore a "double deck" (104 cards) to its original state.

b. How about a 106-card deck?

What mod is it for each of these questions?

Tough Stuff

22. Since a 52-card deck is restored in 8 shuffles, we might expect cards with orbit sizes 1, 2, 4, or 8. But there are no cards with orbit size 4.

Under what conditions will a deck that restores in k shuffles have at least one card for *all* the factors of k as orbit sizes?

23. Without a calculator, determine the number of perfect shuffles it will take to restore a deck of 20,000 cards to its original state.

Oh man. 20,000 cards under the $\mathbb{Z}$? Somebody call Nemo.

24. **a.** Robin claims to have a box with integer dimensions such that all box face diagonals also have integer length. Find a possible set of dimensions for the box, or prove that no such box can exist.

b. Rebecca claims to have a Robin box, but whose the space diagonal (from one corner of the box to the other corner in eye-popping 3D) *also* has integer length. Find a possible set of dimensions for the box, or prove that no such box can exist.

Warning: box may not actually contain Robin or Batman. Box may contain a marmot. Box cannot be used as a cowbell.

Problem Set 10 Table #1

Six perfect shuffles return a 21-card deck back to its original state. So, there are six “transformations” that we can perform on this deck of cards:

- Perfect shuffle the deck 0 times (identity transformation)
- Perfect shuffle the deck 1 time
- Perfect shuffle the deck 2 times
- Perfect shuffle the deck 3 times
- Perfect shuffle the deck 4 times
- Perfect shuffle the deck 5 times

A symmetry is a transformation that leaves something appearing identical. In this case, it’s a transformation that leaves a card in its original position.

Here are the positions of all 21 cards after each transformation.

Card Number	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0 shuffles	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1 shuffle	0	2	4	6	8	10	12	14	16	18	20	1	3	5	7	9	11	13	15	17	19
2 shuffles	0	4	8	12	16	20	3	7	11	15	19	2	6	10	14	18	1	5	9	13	17
3 shuffles	0	8	16	3	11	19	6	14	1	9	17	4	12	20	7	15	2	10	18	5	13
4 shuffles	0	16	11	6	1	17	12	7	2	18	13	8	3	19	14	9	4	20	15	10	5
5 shuffles	0	11	1	12	2	13	3	14	4	15	5	16	6	17	7	18	8	19	9	20	10
Row 1 Size of orbit of this card	1	6																			
Row 2 # of symmetries of this card (incl. identity)			1																		
Row 3 Multiplicative inverse (reciprocal) of the # in row 1																					

Problem Set 10 Table #2

Original																
Rot 0° (Identity)																
Rot 90°																
Rot 180°																
Rot 270°																
Reflect																
Reflect																
Reflect																
Reflect																
Row 1 Size of orbit of this coloring	1		4													
Row 2 # of symmetries of this coloring (incl. identity)			2													
Row 3 Multiplicative inverse of the # in row 1			$\frac{1}{4}$													

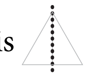
Problem Set 10 Handout #3

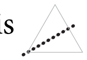
The (uncolored) equilateral triangle has these symmetries:

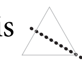
(a) Rotation about the center by 0° ↺ (the identity transformation)

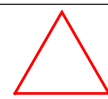
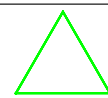
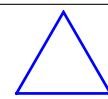
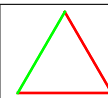
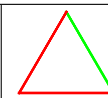
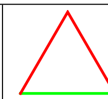
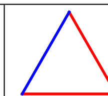
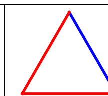
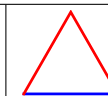
(b) Rotation about the center by 120° ↺

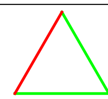
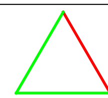
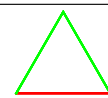
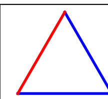
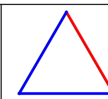
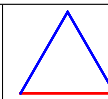
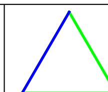
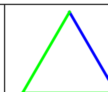
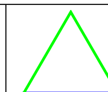
(c) Rotation about the center by 240° ↺

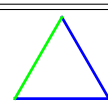
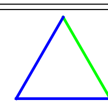
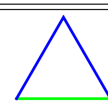
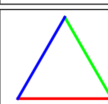
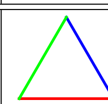
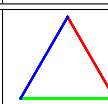
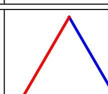
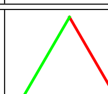
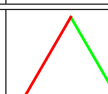
(d) Reflection about this axis 

(e) Reflection about this axis 

(f) Reflection about this axis 

Coloring									
Row 1: Size of orbit of this coloring	1								
Row 2: # of symmetries of this coloring (incl. identity)									
Row 3: Multiplicative inverse of the # in row 1				$\frac{1}{3}$					

Coloring									
Row 1: Size of orbit of this coloring									
Row 2: # of symmetries of this coloring (incl. identity)	2								
Row 3: Multiplicative inverse of the # in row 1									

Coloring									
Row 1: Size of orbit of this coloring									6
Row 2: # of symmetries of this coloring (incl. identity)									
Row 3: Multiplicative inverse of the # in row 1									

Day 11: The Rotary

Opener

1. Kevin has a bunch of congruent non-square rectangles. Because he colors each vertex of each rectangle either corn or cornflower blue, he says that there are 16 different colorings. However, some colorings look identical to others through a re-orientation. Determine the number of "unique" colorings of the rectangles up to re-orientation.

We encourage you to use the index cards to try this out at your tables. When you color the corners of your cards, remember to color both sides.

Important Stuff

2. Complete Handout #1.

Aw jeez, how many handouts this time? Wouldn't you like to know!

- **a.** For each rectangle, what can you say about its orbit size, compared to its number of symmetries?
- **b.** Add up the numbers in the third row. What happens? Can you explain why this works?
- **c.** What do you notice about the numbers in the fourth row?
- **d.** The sum of numbers in the fourth row is $\dfrac{\boxed{}}{4}$.

Hey that's not a question. That's just a weird box. WHAT'S IN THE BOX???

3. Kate says she can calculate the number of unique colorings of an object up to re-orientation by counting all the symmetries among all possible colorings, then dividing by something.

- **a.** What. Does this even work?
- **b.** What is the thing she divides by?
- **c.** Test this theory on one of the handouts from Day 10.

Do not test the theory with string, and definitely do not test the theory with big bangs.

4.
- **a.** Complete Handout #2.
- **b.** How does Row 2 of Handout #1 relate to the bottom row of Handout #2?

What up with that, I say, what up with that?

5. Brian loves Kevin's rectangles, but feels stifled by only having two colors. He wants a palette of 31 colors.

Stifled, shuffled, it's all in the mind . . .

- **a.** If we are unconcerned with re-orientations, how many possible different colorings are there?

Hm, this number might be copyrighted. BR has one of the best logos ever. They don't sell cabinets or frappes, but they will add jimmies for no charge.

b. Now Brian tells you that he is only interested in colorings with 180° rotational symmetry, though it's okay if those colorings have other symmetries too. How many possible colorings are there now, being unconcerned about re-orientations?

Hey, don't forget to use the bubbler at break today!

6. Look at the Ultimate Table. Find an important relationship between the number of units in mod n and the cycle length of 2^k in mod n that is consistently true. Do it again between the number of units in mod n and the cycle length of 10^k in mod n.

Hey wait a minute, this is just Handout #3 in disguise!

Here are some unimportant relationships between the columns:

"The columns all contain numbers."

"The columns all contain positive numbers."

"The columns all contain one-, two-, and three-digit numbers."

"None of the entries in any column is the number 8675309."

Neat Stuff

7. Describe what Handout #2 might have looked like if there were 31 colors instead of 2 colors: How many rows and columns? How many check marks in each row?

8. Zarina points out that even though Brian has a palette of 31 colors, many of the 31^4 different colorings look identical to others through a re-orientation. Determine the number of "unique" colorings of these rectangles up to re-orientation.

9. **a.** Patrick has a cube and paints each face either peach or pink. How many unique colorings are possible? By "unique" we mean that you can't make one look like the other through a re-orientation.

b. Megan also has a cube, but has a palette of three colors to choose from for the faces, instead of two. Estimate how many "unique" colorings there are.

This is also Problem 14 from Day 1. This is also Problem 11 from Day 10. This is also Problem 9 from Day 11. Choosing every face pink or peach is allowed.

This is difficult, so estimate! Don't spend as much time here as you might waiting for a tasty mango dessert.

10. Angela wants to corner the made-to-order market for bracelets with two equispaced beads. Determine the number of unique bracelet designs if she allows for 2 colors. 3 colors? 4 colors? k colors?

11. Nancy's made-to-order bracelet company offers three bead colors to choose from. Watch out, Rob! How many unique bracelet designs can be made if each bracelet has 4 equispaced beads? Five beads?

Equispaced also refers to the positioning of horses at the start of a race.

12. Three Amandas describe three different ways to find multiple rotational symmetries of a cube. Follow along, specifying the axis of rotation and the angle for each symmetry..

a. Amanda tells you to hold the cube with your fingers at the centers of two opposite faces and spin!

b. Amanda tells you to hold the cube with your fingers at two furthest-apart vertices and spin!

c. Amanda tells you to hold the cube with your fingers at the midpoints of two parallel and opposite edges and spin!

d. The descriptions by the three Amandas can help you find all 24 symmetries of the cube. How many symmetries does each Amanda describe?

There are 24 rotational symmetries of the cube, including the identity transformation.

Wanna see this? Go to `go.edc.org/gleamingcube`.

Lookit dem cubes! They’re gleaming! Christian Slater would be proud, but I heard he turned into a robot.

These edges are catty-corner from one another.

13. Build an addition table for $\mathbb{Z}_4$. Oh, that’s just mod 4. Also, build a multiplication table for $\mathbb{U}_5$. Remember $\mathbb{U}_5$ contains all the units in mod 5, which are 1, 2, 4 and 3. What’s with the weird order? That’s word.

$(\mathbb{Z}_4, +)$	0	1	2	3
0				
1				
2				
3				

$(\mathbb{U}_5, \times)$	1	2	4	3
1				
2				
4				
3				

What word? Also what’s with New Englanders using “rotary” and “catty-corner” and “jimmies”? It’s enough to make you want to head to the packie, then sit down with the clicker while eating a fluffernutter and drinking some tonic. Go Sawx.

14. **a.** In $\mathbb{U}_9$ there are six numbers: 1, 2, 4, 5, 7, 8. The number 2 is called a *generator* of $\mathbb{U}_9$ because every number is a power of 2: $1, 2, 4, 8, 7, 5, 1, \ldots$. Which other numbers in $\mathbb{U}_9$ are generators?

b. What are the generators in $\mathbb{Z}_6$ under addition? Instead of using powers (repeated multiplication) of numbers, you will need to use repeated addition.

c. How can generators help you match the tables for $(\mathbb{U}_9, \times)$ and $(\mathbb{Z}_6, +)$?

It tours around $\mathbb{U}_9$, from 1 then all the way
It’s 2, go 2, generate it now
2
That’s all there is to say
$\mathbb{U}$ can’t touch this

15. Do Problem 13 from Day 10 if you haven’t already, then compare to the addition table for $\mathbb{Z}_6$. Can you make those two tables match by pairing the entries in some way? Explain!

16. What are the generators in $\mathbb{Z}_7$ under addition? In $\mathbb{Z}_8$? Neat.

Of course it's neat, this is Neat Stuff.

17. Describe some conditions under which you can say that the tables for two operations can *definitely not* be matched. Find more than one condition!

I told you, homeboy, $\mathbb{U}$ can't match this!

18. Make a table of the number of generators of $\mathbb{U}_n$ for different n. What patterns do you notice?

$\mathbb{U}$ need to calm down.

19. **a.** Find the cycle lengths of 3 and 5 under multiplication in $\mathbb{U}_{13}$ and explain why each is *not* a generator.
b. What are the generators of $\mathbb{U}_{13}$?

Never mind, I'll find someone like $\mathbb{U}$ . . .

20. **a.** Find the number of perfect shuffles that it will take to restore a deck of 90 cards. Do this without a calculator and using an efficient method, given what you learned in Problem 6.
b. Find the number of repeating digits in the decimal for 1/73 without long division or a calculator.

$\mathbb{U}$ might be more efficient if $\mathbb{U}$ look back at Problem 20 from Problem Set 10.

I guess the change in my pocket wasn't enough, I'm like, forget $\mathbb{U}$.

21. Make a multiplication chart for all 24 rotational symmetries of the cube! Mwahahaha!

This problem is quite a grinder! I'm not sure if it has any meatballs, though.

22. Find a number n so that the fraction $\frac{1}{n}$ is a repeating decimal with exactly 8,675,309 digits.

Tough Stuff

23. If p is prime, prove that every $\mathbb{U}_p$ has at least one generator.

thank $\mathbb{U}$, next.

24. If p is prime, prove that every $\mathbb{U}_p$ has exactly ________ generators. Hm, looks like we left that spot blank.

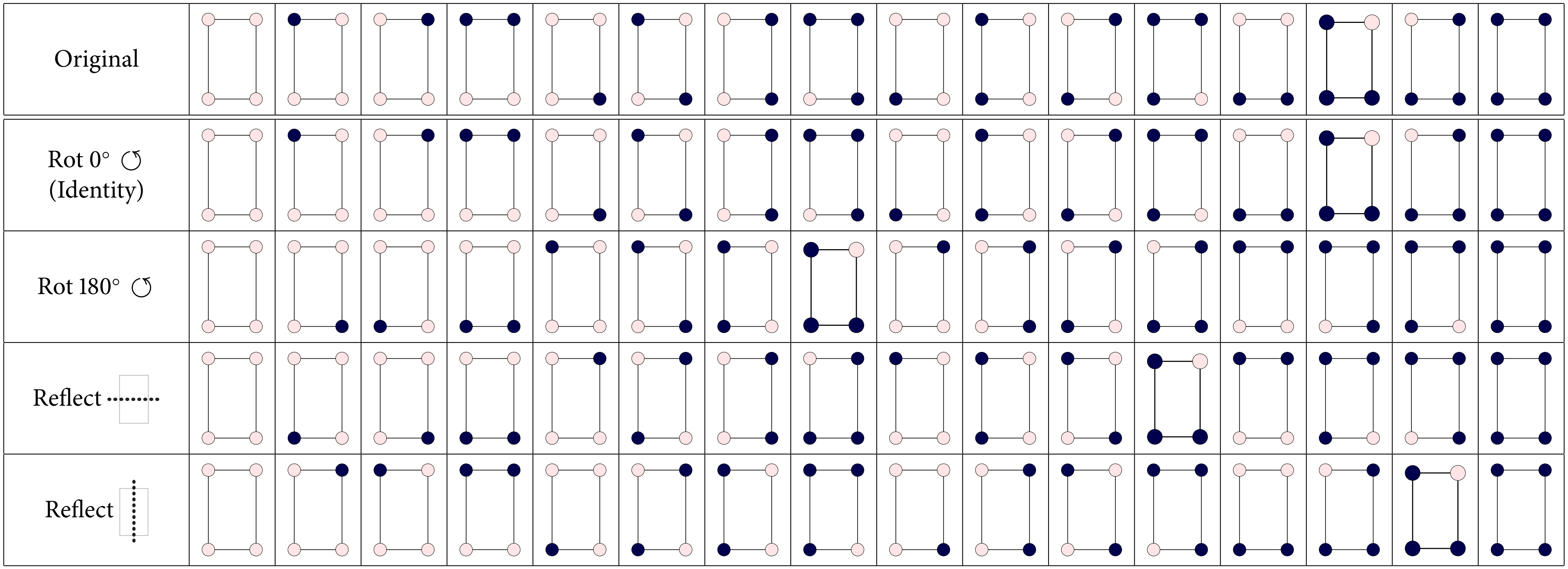
25. Find all *composite* n for which $\mathbb{U}_n$ has generators.

Who are $\mathbb{U}$? Who who, who who?

26. For p prime, find general conditions under which the number 2 is *definitely* or *definitely not* a generator in $\mathbb{U}_p$.

If you missed any of today's comments, look them up on $\mathbb{U}_{\text{tube}}$.

Problem Set 11 Handout #1

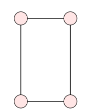
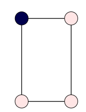
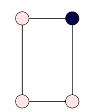
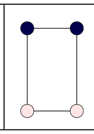
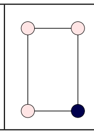
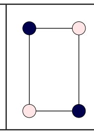
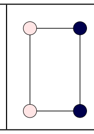
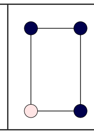
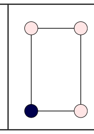
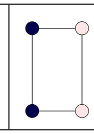
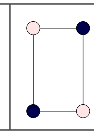
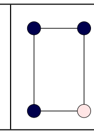
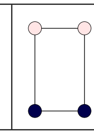
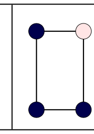
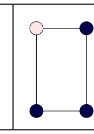
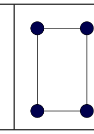


Row 1 Size of orbit of this coloring	1		4													
Row 2 # of symmetries of this coloring (incl. identity)			1													
Row 3 Reciprocal of the # in row 1			$\frac{1}{4}$													
Row 4 The # in row 3 rewritten as...	$\frac{\quad}{4}$	$\frac{\quad}{4}$	$\frac{1}{4}$	$\frac{\quad}{4}$	$\frac{\quad}{4}$	$\frac{\quad}{4}$	$\frac{\quad}{4}$	$\frac{\quad}{4}$	$\frac{\quad}{4}$	$\frac{\quad}{4}$	$\frac{\quad}{4}$	$\frac{\quad}{4}$	$\frac{\quad}{4}$	$\frac{\quad}{4}$	$\frac{\quad}{4}$	$\frac{\quad}{4}$

Problem Set 11 Handout #2

In each white box, determine if the coloring in the column heading has the symmetry in the row heading. Put a check mark in the box if it does. The entire first row has been checked, since every coloring is unchanged if the identity transformation is applied to it.

When you're done with the check marks, count up the total number of check marks in each row and column and write those totals in the shaded "Row Count" and "Column Count" boxes. Write the total number of check marks in the lower-right corner.

Coloring																	Row Count	
Identity	✔	✔	✔	✔	✔	✔	✔	✔	✔	✔	✔	✔	✔	✔	✔	✔	16	or 2^4
Rot 180° ↻																		
Reflect																		
Reflect																		
Column Count																	$\sum =$	

Ultimate Table for Problem Set 11

n	# of units	Cycle length of	
	in mod n	2^k in mod n	10^k in mod n
3	2	2	1
7	6	3	6
9	6	6	1
11	10	10	2
13	12	12	6
17	16	8	16
19	18	18	18
21	12	6	6
23	22	11	22
27	18	18	3
29	28	28	28
31	30	5	15
33	20	10	2
37	36	36	3
39	24	12	6
41	40	20	5
43	42	14	21
47	46	23	46
49	42	21	42
51	32	8	16
53	52	52	13
57	36	18	18
59	58	58	58
61	60	60	60
63	36	6	6
67	66	66	33
69	44	22	22
71	70	35	35
73	72	9	8
77	60	30	6
79	78	39	13
81	54	54	9
83	82	82	41
87	56	28	28
89	88	11	44
91	72	12	6

n	# of units	Cycle length of	
	in mod n	2^k in mod n	10^k in mod n
93	60	10	15
97	96	48	96
99	60	30	2
101	100	100	4
103	102	51	34
107	106	106	53
109	108	36	108
111	72	36	3
113	112	28	112
117	72	12	6
119	96	24	48
121	110	110	22
123	80	20	5
127	126	7	42
129	84	14	21
131	130	130	130
133	108	18	18
137	136	68	8
139	138	138	46
141	92	46	46
143	120	60	6
147	84	42	42
149	148	148	148
151	150	15	75
153	96	24	16
157	156	52	78
159	104	52	13
161	132	33	66
163	162	162	81
167	166	83	166
169	156	156	78
171	108	18	18
173	172	172	43
177	116	58	58
179	178	178	178
181	180	180	180

Day 12: The Table

Opener

5. Distribute the sheets of decimals at your table. Each sheet lists the decimal equivalent for $0/n, 1/n, \ldots, (n-1)/n$ for different n. Each person will look for these two things on their sheet:

- the number of "unique" cycles of digits, and
- the different lengths of repeating decimals.

We consider $\overline{142857}$ and $\overline{428571}$ to be the same cycle, since they use the same digits in the same order. Share your findings with each other. You might find color helpful.

What? **5**? We want you to do these problems in the order they're presented, despite the weird numbering. Honestly we're not sure why they're in this order, but you might be able to make heads of it.

Important Stuff

1. **a.** Complete Handout #1 for Spencer's chessboards. Try to fill in the check marks row-by-row.

b. How many chessboard colorings are "unique" up to re-orientations?

c. Use the handout to help you determine the number of "unique" colorings, if the color of each square of the chessboard can be independently chosen from a palette of 31 colors.

Spencer believes chess is way more fun on a 2-by-2 board. Spencer is wrong.

3. Today's Problem 1, Problem 9 from Day 2, and Problem 1 from Day 11 all involve coloring four things (squares, beads, or corners) and two color choices. Why aren't all the answers the same?

Problem 1? You mean the first problem, right? No. But this is Problem 3, and it's the third problem, right? Yes. Why are you doing this? I don't know, third base.

6. Revisit Problem 1 from Day 11. This time, think of the rectangles as stickers, so you can't flip them over and can only rotate them.

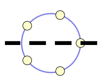
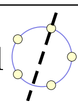
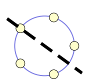
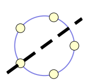
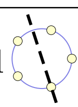
a. How many unique colorings are possible under rotations only?

b. How many unique colorings are possible if you use a palette of 10 colors instead of 2?

2. Trisha is cornering the market on five-bead bracelets. Since she's picked 10 colors for her beads, there are 10^5 possible bracelets if we are unconcerned with re-orientations.

Trisha's 10 colors are: udon know me, schnapps out of it, so many clowns so little time, no more mr. night sky, can't a fjord not to, rice rice baby, tempura-ture rising, pot calling the kettle taupe, machu peach-u, and you're such a buda-pest.

a. Complete this table.

	How many of the 10^5 bracelets have this symmetry?
Rot 0° ⟲ (identity)	
Rot 72° ⟲	
Rot 144° ⟲	
Rot 216° ⟲	
Rot 288° ⟲	
Refl 	
Refl 	
Refl 	
Refl 	
Refl 	

Is this an obnoxious table-spreading table? Maybe a little, but it's got lots of little coicles.

Fun fact about llamas: every llama's last name is Carson.

Ooh, another fun fact about llamas: Hannibal and his army crossed the Alps on llamas to defeat Rome.

b. How many "unique" bracelets are there? A bracelet is "unique" if it cannot be made from any other bracelet through rotations and reflections. Hint: Your answer should be less than 10^5.

This problem is all about pentagons, man. But did you know Pentagon Man also hates Person Man?

12. If Trisha has k colors for each bead, find a formula in terms of k for the number of "unique" bracelets. Verify that your formula works when $k = 10$ and when $k = 2$.

7. Find the number of six-bead bracelets that are unique up to rotations and reflections, if there are k possible colors for each bead.

Neat Stuff

13. **a.** Find the number of bracelets with 7 equispaced beads that are unique up to rotations and reflections, if there are k possible colors for each bead.

b. 8 beads!

So now we've dealt with necklaces of sizes 5, 6, 7, 8. But what about Schlemiel, Schlimazel, and Hasenpfeffer Incorporated?

10. In these bracelet problems, we've allowed reflections. What if the the bracelets are really regular polygons on stickers, so they can't be turned over? How many unique colorings are possible up to rotations only?

14. Real chessboards have 64 squares arranged in a 8×8 grid. Determine how many colorings are unique up to rotations only, if each square is colored one of 10 colors? k colors?

Finally a real chessboard . . . wait, that square looks like it's the color of tempura, and that one looks like a clown. What a budapest!

9. Find the number of bracelets with n equispaced beads that are unique up to rotations and reflections, if there are k possible colors for each bead. Your answer will depend on whether n is even or odd.

4. If you haven't already done Problem 12 from Day 11 and seen `go.edc.org/gleamingcube`, please do so now. Make sure you can enumerate all 24 rotational symmetries of the cube.

Each animation shows *one* of several possible versions of that type of symmetry. To make the animation more interesting, a cowbell will be hit every time the cube rotates into a symmetry. What? The cowbell player got sick after one too many games of pickleball? Oh, one too many pickle*backs*, I see.

8. Alex wants to color each face of a cube from a palette of k colors. Not considering rotations of the cube, there are k^6 different colorings possible.

a. How many of those k^6 colorings have Amanda #1's 90° rotational symmetry?

b. How many of those k^6 colorings have Amanda #1's 180° rotational symmetry?

c. How many of those k^6 colorings have Amanda #2's 120° rotational symmetry?

d. How many of those k^6 colorings have Amanda #3's 180° rotational symmetry?

e. Use this information to determine the number of colorings of the faces of a cube that are unique up to rotations. Check that your answer matches your previous work for $k = 2$.

Spoiler alert! Do not turn the page! There is a monstrous function at the end of this problem set!

11. Gen stealthily writes down the function

$$f(k) = \frac{1}{24}\left(k^6 + 3k^4 + 12k^3 + 8k^2\right).$$

Calculate $f(2)$ and $f(3)$.

Well, look at that! I, loveable, furry old Grover am the monster at the end of this problem set. And you were so scared.

15. A d6 has the numbers 1 through 6 on it. How many "unique" ways are there to put the numbers on a d6? Here, we mean unique up to rotations.

Thank goodness, the problem numbers are back to normal now. I can't imagine what sort of message it sends to have 14 numbers put in a specific order like that.

16. Ishrat and Max encourage you to go back to Handout #2 from Day 11 and try to "count vertically" instead of horizontally in the case of 31 colors.

Never heard of a d6? It's also known as a *number cube*. The state motto of New Hampshire is "Live Free Or Number Cube". One of the worst James Bond movies was "Number Cube Another Day", wbich was much worse than "Live And Let Number Cube". None of these movies starred Vin Number Cubesel, but Vin has been to Sundance to see some low-budget innumber cube films. The d6 is also a significant part of the Keto Number Cubet. If you're part of a Cutting Crew, you can sing "I Just Number Cubed In Your Arms Tonight". But if you prefer classical music, perhaps Mozart's "Number Cubes Irae" is more your style. Thanks, thanks, you've been a terrific aunumber cubence.

17. Gareth wants to color each *vertex* of a cube from a palette of k colors. How many colorings are unique up to rotations?

18. Jane wants to color each *edge* of a cube from a palette of k colors. How many colorings are unique up to rotations?

19. An element of a group is a *generator* if repeated operation of that element takes you through every element of the group.

- **a.** Find all the generators for $(\mathbb{Z}_{12}, +)$ or explain why there aren't any.
- **b.** Find all the generators for $(\mathbb{U}_{12}, \times)$ or explain why there aren't any.

20. Sometimes $(\mathbb{U}_n, \times)$ has a generator, sometimes it don't.

- **a.** Under what conditions will $\mathbb{U}_n$ have a generator?
- **b.** In terms of n, *how many generators* are there?

Sometimes you feel like a generator, sometimes you don't.

Tough Stuff

21. How many "unique" ways are there to color 3 vertices of an icosahedron white, while coloring all other vertices black?

22. How many non-isomorphic graphs with n vertices are there? And what does this question even mean?

0/23	0.0		
1/23	$0.\overline{0434782608695652173913}$	12/23	$0.\overline{5217391304347826086956}$
2/23	$0.\overline{0869565217391304347826}$	13/23	$0.\overline{5652173913043478260869}$
3/23	$0.\overline{1304347826086956521739}$	14/23	$0.\overline{6086956521739130434782}$
4/23	$0.\overline{1739130434782608695652}$	15/23	$0.\overline{6521739130434782608695}$
5/23	$0.\overline{2173913043478260869565}$	16/23	$0.\overline{6956521739130434782608}$
6/23	$0.\overline{2608695652173913043478}$	17/23	$0.\overline{7391304347826086956521}$
7/23	$0.\overline{3043478260869565217391}$	18/23	$0.\overline{7826086956521739130434}$
8/23	$0.\overline{3478260869565217391304}$	19/23	$0.\overline{8260869565217391304347}$
9/23	$0.\overline{3913043478260869565217}$	20/23	$0.\overline{8695652173913043478260}$
10/23	$0.\overline{4347826086956521739130}$	21/23	$0.\overline{9130434782608695652173}$
11/23	$0.\overline{4782608695652173913043}$	22/23	$0.\overline{9565217391304347826086}$

0/31	0.0		
1/31	$0.\overline{032258064516129}$	16/31	$0.\overline{516129032258064}$
2/31	$0.\overline{064516129032258}$	17/31	$0.\overline{548387096774193}$
3/31	$0.\overline{096774193548387}$	18/31	$0.\overline{580645161290322}$
4/31	$0.\overline{129032258064516}$	19/31	$0.\overline{612903225806451}$
5/31	$0.\overline{161290322580645}$	20/31	$0.\overline{645161290322580}$
6/31	$0.\overline{193548387096774}$	21/31	$0.\overline{677419354838709}$
7/31	$0.\overline{225806451612903}$	22/31	$0.\overline{709677419354838}$
8/31	$0.\overline{258064516129032}$	23/31	$0.\overline{741935483870967}$
9/31	$0.\overline{290322580645161}$	24/31	$0.\overline{774193548387096}$
10/31	$0.\overline{322580645161290}$	25/31	$0.\overline{806451612903225}$
11/31	$0.\overline{354838709677419}$	26/31	$0.\overline{838709677419354}$
12/31	$0.\overline{387096774193548}$	27/31	$0.\overline{870967741935483}$
13/31	$0.\overline{419354838709677}$	28/31	$0.\overline{903225806451612}$
14/31	$0.\overline{451612903225806}$	29/31	$0.\overline{935483870967741}$
15/31	$0.\overline{483870967741935}$	30/31	$0.\overline{967741935483870}$

0/53	0.0		
1/53	$0.\overline{0188679245283}$	27/53	$0.\overline{5094339622641}$
2/53	$0.\overline{0377358490566}$	28/53	$0.\overline{5283018867924}$
3/53	$0.\overline{0566037735849}$	29/53	$0.\overline{5471698113207}$
4/53	$0.\overline{0754716981132}$	30/53	$0.\overline{5660377358490}$
5/53	$0.\overline{0943396226415}$	31/53	$0.\overline{5849056603773}$
6/53	$0.\overline{1132075471698}$	32/53	$0.\overline{6037735849056}$
7/53	$0.\overline{1320754716981}$	33/53	$0.\overline{6226415094339}$
8/53	$0.\overline{1509433962264}$	34/53	$0.\overline{6415094339622}$
9/53	$0.\overline{1698113207547}$	35/53	$0.\overline{6603773584905}$
10/53	$0.\overline{1886792452830}$	36/53	$0.\overline{6792452830188}$
11/53	$0.\overline{2075471698113}$	37/53	$0.\overline{6981132075471}$
12/53	$0.\overline{2264150943396}$	38/53	$0.\overline{7169811320754}$
13/53	$0.\overline{2452830188679}$	39/53	$0.\overline{7358490566037}$
14/53	$0.\overline{2641509433962}$	40/53	$0.\overline{7547169811320}$
15/53	$0.\overline{2830188679245}$	41/53	$0.\overline{7735849056603}$
16/53	$0.\overline{3018867924528}$	42/53	$0.\overline{7924528301886}$
17/53	$0.\overline{3207547169811}$	43/53	$0.\overline{8113207547169}$
18/53	$0.\overline{3396226415094}$	44/53	$0.\overline{8301886792452}$
19/53	$0.\overline{3584905660377}$	45/53	$0.\overline{8490566037735}$
20/53	$0.\overline{3773584905660}$	46/53	$0.\overline{8679245283018}$
21/53	$0.\overline{3962264150943}$	47/53	$0.\overline{8867924528301}$
22/53	$0.\overline{4150943396226}$	48/53	$0.\overline{9056603773584}$
23/53	$0.\overline{4339622641509}$	49/53	$0.\overline{9245283018867}$
24/53	$0.\overline{4528301886792}$	50/53	$0.\overline{9433962264150}$
25/53	$0.\overline{4716981132075}$	51/53	$0.\overline{9622641509433}$
26/53	$0.\overline{4905660377358}$	52/53	$0.\overline{9811320754716}$

0/57	0.0		
1/57	$0.\overline{017543859649122807}$	29/57	$0.\overline{508771929824561403}$
2/57	$0.\overline{035087719298245614}$	30/57	$0.\overline{526315789473684210}$
3/57	$0.\overline{052631578947368421}$	31/57	$0.\overline{543859649122807017}$
4/57	$0.\overline{070175438596491228}$	32/57	$0.\overline{561403508771929824}$
5/57	$0.\overline{087719298245614035}$	33/57	$0.\overline{578947368421052631}$
6/57	$0.\overline{105263157894736842}$	34/57	$0.\overline{596491228070175438}$
7/57	$0.\overline{122807017543859649}$	35/57	$0.\overline{614035087719298245}$
8/57	$0.\overline{140350877192982456}$	36/57	$0.\overline{631578947368421052}$
9/57	$0.\overline{157894736842105263}$	37/57	$0.\overline{649122807017543859}$
10/57	$0.\overline{175438596491228070}$	38/57	$0.\overline{6}$
11/57	$0.\overline{192982456140350877}$	39/57	$0.\overline{684210526315789473}$
12/57	$0.\overline{210526315789473684}$	40/57	$0.\overline{701754385964912280}$
13/57	$0.\overline{228070175438596491}$	41/57	$0.\overline{719298245614035087}$
14/57	$0.\overline{245614035087719298}$	42/57	$0.\overline{736842105263157894}$
15/57	$0.\overline{263157894736842105}$	43/57	$0.\overline{754385964912280701}$
16/57	$0.\overline{280701754385964912}$	44/57	$0.\overline{771929824561403508}$
17/57	$0.\overline{298245614035087719}$	45/57	$0.\overline{789473684210526315}$
18/57	$0.\overline{315789473684210526}$	46/57	$0.\overline{807017543859649122}$
19/57	$0.\overline{3}$	47/57	$0.\overline{824561403508771929}$
20/57	$0.\overline{350877192982456140}$	48/57	$0.\overline{842105263157894736}$
21/57	$0.\overline{368421052631578947}$	49/57	$0.\overline{859649122807017543}$
22/57	$0.\overline{385964912280701754}$	50/57	$0.\overline{877192982456140350}$
23/57	$0.\overline{403508771929824561}$	51/57	$0.\overline{894736842105263157}$
24/57	$0.\overline{421052631578947368}$	52/57	$0.\overline{912280701754385964}$
25/57	$0.\overline{438596491228070175}$	53/57	$0.\overline{929824561403508771}$
26/57	$0.\overline{456140350877192982}$	54/57	$0.\overline{947368421052631578}$
27/57	$0.\overline{473684210526315789}$	55/57	$0.\overline{964912280701754385}$
28/57	$0.\overline{491228070175438596}$	56/57	$0.\overline{982456140350877192}$

0/63	0.0		
1/63	$0.\overline{015873}$	32/63	$0.\overline{507936}$
2/63	$0.\overline{031746}$	33/63	$0.\overline{523809}$
3/63	$0.\overline{047619}$	34/63	$0.\overline{539682}$
4/63	$0.\overline{063492}$	35/63	$0.\overline{5}$
5/63	$0.\overline{079365}$	36/63	$0.\overline{571428}$
6/63	$0.\overline{095238}$	37/63	$0.\overline{587301}$
7/63	$0.\overline{1}$	38/63	$0.\overline{603174}$
8/63	$0.\overline{126984}$	39/63	$0.\overline{619047}$
9/63	$0.\overline{142857}$	40/63	$0.\overline{634920}$
10/63	$0.\overline{158730}$	41/63	$0.\overline{650793}$
11/63	$0.\overline{174603}$	42/63	$0.\overline{6}$
12/63	$0.\overline{190476}$	43/63	$0.\overline{682539}$
13/63	$0.\overline{206349}$	44/63	$0.\overline{698412}$
14/63	$0.\overline{2}$	45/63	$0.\overline{714285}$
15/63	$0.\overline{238095}$	46/63	$0.\overline{730158}$
16/63	$0.\overline{253968}$	47/63	$0.\overline{746031}$
17/63	$0.\overline{269841}$	48/63	$0.\overline{761904}$
18/63	$0.\overline{285714}$	49/63	$0.\overline{7}$
19/63	$0.\overline{301587}$	50/63	$0.\overline{793650}$
20/63	$0.\overline{317460}$	51/63	$0.\overline{809523}$
21/63	$0.\overline{3}$	52/63	$0.\overline{825396}$
22/63	$0.\overline{349206}$	53/63	$0.\overline{841269}$
23/63	$0.\overline{365079}$	54/63	$0.\overline{857142}$
24/63	$0.\overline{380952}$	55/63	$0.\overline{873015}$
25/63	$0.\overline{396825}$	56/63	$0.\overline{8}$
26/63	$0.\overline{412698}$	57/63	$0.\overline{904761}$
27/63	$0.\overline{428571}$	58/63	$0.\overline{920634}$
28/63	$0.\overline{4}$	59/63	$0.\overline{936507}$
29/63	$0.\overline{460317}$	60/63	$0.\overline{952380}$
30/63	$0.\overline{476190}$	61/63	$0.\overline{968253}$
31/63	$0.\overline{492063}$	62/63	$0.\overline{984126}$

0/77	0.0		
1/77	$0.\overline{012987}$	39/77	$0.\overline{506493}$
2/77	$0.\overline{025974}$	40/77	$0.\overline{519480}$
3/77	$0.\overline{038961}$	41/77	$0.\overline{532467}$
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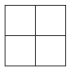
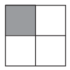
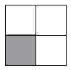
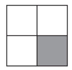
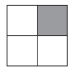
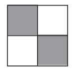
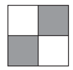
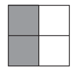
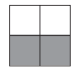
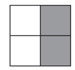
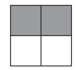
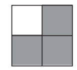
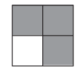
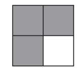
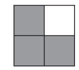
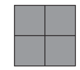
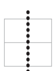
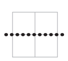
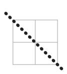
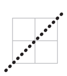
5/77	$0.\overline{064935}$	43/77	$0.\overline{558441}$
6/77	$0.\overline{077922}$	44/77	$0.\overline{571428}$
7/77	$0.\overline{09}$	45/77	$0.\overline{584415}$
8/77	$0.\overline{103896}$	46/77	$0.\overline{597402}$
9/77	$0.\overline{116883}$	47/77	$0.\overline{610389}$
10/77	$0.\overline{129870}$	48/77	$0.\overline{623376}$
11/77	$0.\overline{142857}$	49/77	$0.\overline{63}$
12/77	$0.\overline{155844}$	50/77	$0.\overline{649350}$
13/77	$0.\overline{168831}$	51/77	$0.\overline{662337}$
14/77	$0.\overline{18}$	52/77	$0.\overline{675324}$
15/77	$0.\overline{194805}$	53/77	$0.\overline{688311}$
16/77	$0.\overline{207792}$	54/77	$0.\overline{701298}$
17/77	$0.\overline{220779}$	55/77	$0.\overline{714285}$
18/77	$0.\overline{233766}$	56/77	$0.\overline{72}$
19/77	$0.\overline{246753}$	57/77	$0.\overline{740259}$
20/77	$0.\overline{259740}$	58/77	$0.\overline{753246}$
21/77	$0.\overline{27}$	59/77	$0.\overline{766233}$
22/77	$0.\overline{285714}$	60/77	$0.\overline{779220}$
23/77	$0.\overline{298701}$	61/77	$0.\overline{792207}$
24/77	$0.\overline{311688}$	62/77	$0.\overline{805194}$
25/77	$0.\overline{324675}$	63/77	$0.\overline{81}$
26/77	$0.\overline{337662}$	64/77	$0.\overline{831168}$
27/77	$0.\overline{350649}$	65/77	$0.\overline{844155}$
28/77	$0.\overline{36}$	66/77	$0.\overline{857142}$
29/77	$0.\overline{376623}$	67/77	$0.\overline{870129}$
30/77	$0.\overline{389610}$	68/77	$0.\overline{883116}$
31/77	$0.\overline{402597}$	69/77	$0.\overline{896103}$
32/77	$0.\overline{415584}$	70/77	$0.\overline{90}$
33/77	$0.\overline{428571}$	71/77	$0.\overline{922077}$
34/77	$0.\overline{441558}$	72/77	$0.\overline{935064}$
35/77	$0.\overline{45}$	73/77	$0.\overline{948051}$
36/77	$0.\overline{467532}$	74/77	$0.\overline{961038}$
37/77	$0.\overline{480519}$	75/77	$0.\overline{974025}$
38/77	$0.\overline{493506}$	76/77	$0.\overline{987012}$

Problem Set 12 Handout #1

In each white box, determine if the coloring in the column heading has the symmetry in the row heading. Put a check mark in the box if it does. The entire first row has been checked, since every coloring is unchanged if the identity transformation is applied to it.

When you're done with the check marks, count up the total number of check marks in each row and column and write those totals in the shaded "Row Count" and "Column Count" boxes. Write the total number of check marks in the lower-right corner.

Coloring																	Row Count	
Identity	✔	✔	✔	✔	✔	✔	✔	✔	✔	✔	✔	✔	✔	✔	✔	✔	16	or 2^4
Rot 90° ↺																		
Rot 180° ↺																		
Rot 270° ↺																		
Reflect 																		
Reflect 																		
Reflect 																		
Reflect 																		
Column Count																	$\sum =$	

Day 13: The Chain

Important Stuff

1. **a.** What is the decimal expansion for $\frac{1}{39}$?

b. Hey! Watch this: `http://go.edc.org/40cards`. There are ten piles; what pile is card #1 in before each shuffle?

Yo! Where my opener at! Check the message on the projectors.

c. Use the piles to find the decimal expansion for $\frac{2}{39}$ and $\frac{13}{39}$.

d. Can you explain why this works?

2. **a.** What are the six powers of 10 in mod 77?

b. One of the charts from Day 12 included decimal expansions of $n/77$. What do you notice about the decimal expansions for these six fractions:

$$\frac{1}{77}, \frac{10}{77}, \frac{23}{77}, \frac{76}{77}, \frac{67}{77}, \frac{54}{77}$$

Why does that happen?

c. Find all *eleven* solutions of the equation

$$22x = 0 \qquad \text{in mod } 77.$$

This equation says "22 multiplied by what number is a multiple of 77?". Bowen likes 77 because it is $7 \cdot 11$.

d. Find all *some number of* solutions of the equation

$$100x = x \qquad \text{in mod } 77.$$

100 and 99 are both too big for mod 77, so make them smaller! (This was a rejected Carly Rae lyric.)

e. Which $n/77$ have a repeating decimal length of 2?

3. The decimal expansions for the fractions $\frac{1}{81}, \frac{2}{81}, \ldots \frac{80}{81}$ exhibit repeating decimals with lengths 1, 3, and 9.

a. For which n will $n/81$ have repeating decimals of length 1? 3? 9?

b. Find all solutions of the equation

$$1000x = x \qquad \text{in mod } 81.$$

Since 1000 and 999 are too big, you can "reduce" them into mod 81 first. If it's helpful. And it probably is.

4. The decimal expansions for the fractions $\frac{1}{143}, \frac{2}{143}, \ldots \frac{142}{143}$ exhibit repeating decimals with lengths 2 and 6.

a. Find all solutions of the equation

$$100x = x \qquad \text{in mod } 143.$$

This problem brought to you by Brooklyn Nine-Nine, now on NBC! 99 and 143 are both multiples of . . .

b. Find all solutions of the equation

$$1000x = x \qquad \text{in mod } 143.$$

. . . that previous note brought to you by Stranger Things, home of Eleven!

5. Complete this table.

This box presented by the number 143! Now with the flavor of twin primes! Why not use the favorite number of Mr. Rogers today? When you need to say "I love you", say it with 143!

	How many integers from 0 to 142 make this equation true in mod 143?
$10^0x = x$	
$10^1x = x$	
$10^2x = x$	
$10^3x = x$	
$10^4x = x$	
$10^5x = x$	

How could you use this table to calculate the number of unique cycles of repeating digits are present in the decimal expansions of $\frac{0}{143}, \frac{1}{143}, \ldots, \frac{142}{143}$?

I think we're supposed to count some symmetries here . . .

6. Use the method above to find the number of unique cycles for some of the sets of decimal expansions that you looked at on Day 12.

7. Change the method above to find the number of unique cycles for some of the decks of cards you've looked at previously.

Remember, if you want to work with a 52-card deck, it's controlled by mod 51, not mod 52.

Review Your Stuff

8. We traditionally set aside part of the last problem set for review. Work as a group at your table to write **one** review question for tomorrow's problem set. Spend **at most 20 minutes** on this. Make sure your question is something that $\boxed{\ast\textbf{everyone}\ast}$ at your table can do, and that you expect $\boxed{\ast\textbf{everyone}\ast}$ in the class to be able to do. Problems that connect different ideas we've visited are especially welcome. We reserve the right to use, not use, or edit your questions, depending on how much other material we write, whether you've written your question on the approved piece of paper, your group's ability to write a good joke, how good your bribes are, and hundreds of other factors.

Imagine yourself writing an Important Stuff question, that's what we are looking for here. You got this!

Remember that one time at math camp where you wrote a really bad joke for the problem set? No? Good.

If you need any card shuffling animations, see `go.edc.org/2piles` or `go.edc.org/3piles`.

Neat Stuff

9. If b and d are primes, how many units are in each mod (in terms of b and d)?

Mod *bee*?

- **a.** mod b
- **b.** mod bd
- **c.** mod b^2
- **d.** mod b^2d

10. Given n and its prime factorization, how many units are in mod n?

11. Amy wants to corner the made-to-order market for bracelets with three equispaced beads. Determine the number of unique bracelet designs (up to rotations and reflections) if she allows for 2 colors. 3 colors? 4 colors? k colors?

Did anyone see the triple play by the Bees last night? I think that's what this problem is about. Oh, *beads*!

12. If you're hankering for a general formula for the number of unique bracelet designs for n equispaced beads and k colors, then you might enjoy making this table.

I've got a hanker, chief!

	How many bracelets have this symmetry?				
	$n=3$	$n=4$	$n=5$	$n=6$	$\cdots$
Rot 0 sides ↺					
Rot 1 side ↺					
Rot 2 sides ↺					
Rot 3 sides ↺					
Rot 4 sides ↺					
Rot 5 sides ↺					
$\vdots$					

Here, "Rot p sides ↺" refers to the counterclockwise rotation of the regular polygon by $360° \cdot p/n$.

13. Enumerate the 12 rotational symmetries of a regular tetrahedron.

14. **a.** Bill wants to color each face of a regular tetrahedron from a palette of k colors. How many colorings are unique up to rotations?

b. Jasper wants to color each *vertex* of a regular tetrahedron from a palette of k colors. How many?

What's the official insect of Utah? You would think, but it's actually the aphid.

c. Westley wants to color each *edge* of a regular tetrahedron from a palette of k colors. What now!

Just kidding. Of course it's a bee!

15. Suppose you have a geometric object and a group of symmetries of that object. If Σ means to add up things, then this shtuff and shtuff below is true.

$$\begin{aligned}
\text{\# of ``unique'' colorings of that object up to symmetries} &= \sum_{\text{all colorings}} \frac{1}{\text{size of orbit of that coloring}} \\
&= \sum_{\text{all colorings}} \frac{\text{\# of symmetries of that coloring}}{\text{\# of symmetries in the group}} \\
&= \frac{1}{\text{\# of symmetries in the group}} \sum_{\text{all colorings}} \left(\begin{array}{c}\text{\# of symmetries} \\ \text{of that coloring}\end{array} \right) \\
&= \frac{1}{\text{\# of symmetries in the group}} \sum_{\text{all symmetries}} \left(\begin{array}{c}\text{\# of colorings that} \\ \text{have that symmetry}\end{array} \right)
\end{aligned}$$

There ain't no reason A and B should be alone

Today, yeah baby, today, yeah baby

I'm on the vertex of glory

And I'm hanging on a corner with you

I'm on the vertex, the vertex, the vertex, the vertex, the vertex, the vertex, the vertex

I'm on the vertex of glory

And I'm hanging on a corner with you! I'm on the corner with you!

What does all of this mean and what does it have to do with "unique" bracelets, chessboards, repeating decimals, and card positions during perfect shuffles? Justify each equal sign above.

"What does it all mean, man?" Be sure to say that using George Carlin's voice, without the seven words please.

Tough Stuff

16. **a.** A d20 is labeled with the numbers 1 through 20, one on each of the faces of a regular icosahedron. How many "unique" d20 options are there?

b. Players would object to a d20 if opposite numbers weren't on opposite faces. The 1 needs to be directly opposite the 20. How many "unique" d20 arrangements are there given this restriction?

What? You'd think you would want the 20 right next to the 1 for maximum suspense.

17. Sicherman dice are a *different* way to populate 2d6 with positive integers, so that the sums of the two d6 matched the usual distribution. Only positive integers are allowed, and repetition is allowed.

2d6 means two six-sided dice.

a. What numbers are on the Sicherman dice?

b. Find all possible "Sicherman-like" dice for the d4, d8, d12, and d20. There may be more than one possible answer, or none at all! Woo hoo ha ha ha.

Day 14: The See

Opener

The See? What sort of weird title is that? You'll have to look back on the titles of this week's problem sets to find a connection. Let us know what you saw.

1. **a.** What are the eight powers of 2 in mod 51?

   b. One of the charts from Day 9 included base 2 "decimal" expansions of $n/51$. What do you notice about the base 2 "decimal" expansions for these eight fractions:

$$\frac{1}{51}, \frac{2}{51}, \frac{4}{51}, \frac{8}{51}, \frac{16}{51}, \frac{32}{51}, \frac{13}{51}, \frac{26}{51}$$

   Why does that happen?

Also, did you discover the secret message? Set 12's weird problem ordering and sidenotes might help you.

   c. Find all *three* solutions of the equation

$$3x = 0 \qquad \text{in mod } 51.$$

   d. Find all *whatever number of* solutions of the equation

$$64x = x \qquad \text{in mod } 51.$$

64 is too big, so you might start by making it smaller. Did you know that 525,600 is the same as 64 in mod 51? I hope not, because it's not true.

   e. Which $\frac{n}{51}$ have a repeating base 2 "decimal" of length 2?

Important Stuff

2. Remember `http://go.edc.org/40cards` from the last problem set? Cal tells you to investigate card #13. What's going on?

Sit! Sit! Sit! Sit!

3. A 341-card deck undergoes perfect shuffling. We know, it's a lot to take in.

But does it undergo perfect *Cupid* shuffling? Boy, that song sure says Cupid a lot.

   a. What is the smallest integer n so that $2^n - 1$ is a multiple of 341?

   b. Can you explain why your answer to part a implies that a 341-card deck restores in 10 shuffles?

   c. It would be difficult to directly find the number of unique cycles of the 341-card deck as it undergoes shuffling. So, let's use its 10 "symmetries": shuffle 0 times, shuffle 1 time, . . . , shuffle 9 times! Complete this table to figure this out. Wait, where's the table? Hm, it's around here somewhere.

Yesterday it was 143, now it's 341? Crazy. 341 is also a product of two primes, but not twin primes, and in text it means "You Love I". That usually leads to a bad-grammar breakup.

	How many solutions in mod 341?
$2^0x = x$	
$2^1x = x$	
$2^2x = x$	
$2^3x = x$	
$2^4x = x$	
$2^5x = x$	
$2^6x = x$	
$2^7x = x$	
$2^8x = x$	
$2^9x = x$	

The values in this table grow and shrink faster than the speed of that version of Jingle Bells. Did you know 341 is $31 \cdot 11$? Oh hey, that's true!

d. Now, use the method from Day 13 to show that the 341-card deck has 38 "unique" cycles.

The solution is the same as the number of games played by each Premier League team each season. Fine, not helpful? It's also the number of times Daniel sang hallelujah.

4. If you did Problems 6 and 9 from Day 4 or Problems 10 and 11 from Day 5, use the method of the previous problem to verify the number of unique cycles in the 15-card and 21-card decks.

5. How many unique cycles of repeating digits are present in the decimal expansions of $\frac{0}{1507}, \frac{1}{1507}, \ldots, \frac{1506}{1507}$? In case you were wondering,

$$\frac{1}{1507} = 0.\overline{00066357}$$

Stuff Behind the Stuff

6. **a.** Why is doing perfect shuffles (into 2 piles) with 40 cards essentially the same as with 39 cards?

b. Why do perfect shuffles into n piles involve multiplying by n in some mod?

c. How can you determine the minimum number of perfect shuffles required to restore a deck of n cards, without actually performing the shuffles?

This week on Behind the Stuff: Katy Perry's decision to retitle "Left Shark" to the ultimately more popular "Firework".

And without having someone else perform the shuffles . . .

7. Anne and Kate are looking at expressing a whole number and a fraction into different bases. They are comparing Problem 5 from Day 6 and Problem 3 from Day 7. What is the same and different between those two problems? Why?

Like Eat It versus Beat It, they are mostly pretty similar.

8. What happens when a number in base 10 is multiplied by 10? Table 7 wants to encourage everyone (including us) to consider replacing the phrase "shift decimal point" with "every digit changes its place value." Discuss why might this better support students' thinking.

9. What's the formal reasoning behind the method we used to count the number of unique colorings of an object up to a group of symmetries? See Problem 15 from Day 13.

Formal reasoning wears a cummerbund.

10. **a.** The numbers on a deck of cards are like the colors of beads on a necklace. Discuss.

b. Out of all of the possible rearrangements (transformations) of a deck of n cards, the perfect shuffle generates a special group of rearrangements that cycles back to zero perfect shuffles (the identity transformation). Discuss.

Float like a ten of hearts, sting like a bead. *Bead?*

. . . Yes it does!

Your Stuff

Table 1 Monica decides to start a cube necklace business. Her first necklace design features a necklace with four rotating cubes that can move around the necklace. The sides of the cube that the necklace chain goes through do not show or matter. She decides to use her four favorite colors: Bowen Blue, Chartruse, Razzmatazz and Darryl Strawberry. How many unique necklaces can she wear?

Your jokes. *Our jokes.*

Monica's cube necklaces are gleaming!

For all you New Yorkers out there... You're welcome. DAAAAAAARRRYYLLL!

Table 2 Liz loves to play Pinochle. A Pinochle deck has 48 cards, 9 through A, in each suit represented twice. Her brand new deck of cards come in this order:

$$\begin{array}{ccccccccc} 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & \dots \\ \diamondsuit & \diamondsuit & \heartsuit & \heartsuit & \clubsuit & \clubsuit & \spadesuit & \spadesuit & \end{array}$$

a. How many perfect shuffles will it take to get the cards back to this specific order? Perfect shuffles into 3 piles?

b. Rob plays double deck Pinochle, which means there are two decks combined like this:

$$\begin{array}{cccccc} 9 & 9 & 9 & 9 & 9 & \dots \\ \diamondsuit & \diamondsuit & \diamondsuit & \diamondsuit & \heartsuit & \end{array}$$

Rob also likes double-decker buses, upper-decker home runs, retired wide receiver Eric Decker, and tools from a specific company.

how many perfect shuffles will it take to get the cards back to this specific order? Perfect shuffles into 3 piles?

c. Noel likes Jokers and always puts one as the top card. How many perfect shuffles are needed now?

d. With or without Jokers, can you come up with an ordering of the deck that requires fewer perfect shuffles to return to? What if you can also change the deck size?

Table 3 **a.** How could you do Bowen's magic trick with a 10-pile shuffle of 40 cards? Is it possible?

b. How would the answer change if you had a 5-pile shuffle with a 40-card deck? Explain the process, you don't need to explain the exact calculations.

How could you, after all we've been through together!?! Also, did you talk to Table 7?

Table 4 **a.** Do fractions exist in modular arithmetic? Can you reduce them?

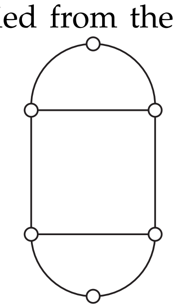
b. Under what conditions can you find solutions to these "mod" equations?

(i) $4x = 1$ in mod 9	(iv) $3x = 5$ in mod 9
(ii) $8x = 2$ in mod 9	(v) $\frac{1}{4} = x$ in mod 9
(iii) $7x = 4$ in mod 9	(vi) $\frac{1}{2} = x$ in mod 51

c. How does your answer to (vi) above relate to card shuffling for a 52-card deck?

Do mother functions exist in modular arithmetic? Please say yes.

Table 5 PCMI's planning committee for 2019–2020 is considering a double-sided reversible name tag so that participants can be identified from the front or the back.



a. PCMI is offering 6 locations for swag tag customization, indicated by the empty circles above. If there are 2 swag tags "Teaching is Lit" and "Mother functions!", how many unique name tag designs are possible up to re-orientations?

Starts on the inside, ya dig?

TI has a song called My Swag.

I'm chasin' that paper, baby, however it come.

b. What if Bowen takes over the swag tag maker and prints out k different swag tags?

c. What swag tags should PCMI use for next summer?

It's a takeover! But never for the front side. Bowen got back.

Table 6 How many shuffles does it take for 85 cards split into 5 piles to return to its original order? What are the unique cycles for this situation?

Table 7 **a.** How do you do perfect shuffles for an arbitrary deck size when you split it into 5 piles? What about n piles?

b. Do Bowen's card trick for 40 cards shuffled into 10 piles in your head!

c. 26 and 1/26 are multiplicative inverses in base 10. Are they multiplicative inverses in base 2?

You can have a pile of 26 cards or a pile of 4 cards, but what do you call a pile of kittens? A MEOW-ntain!

An inverse joke is like a normal joke, but the punchline comes first.

Table 8 Spencer is now coloring a 2×3 chessboard.

a. How many symmetries does an uncolored 2×3 chessboard have?

b. How many unique (up to re-orientations) colorings of the chessboard exist using 2 colors?

c. Using k colors?

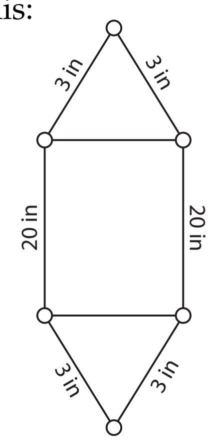
What do baby parabolas drink? Quadratic formula!

Table 9 Let's perfectly shuffle a 36-card deck! Zarina splits the deck into 2 piles while shuffling. Alissa splits the deck into 3 piles. Each person should choose a card number to follow for Zarina's process or Alissa's process. What stays the same? What is different?

1. Bowen was a contestant on *Who Wants to be a Millionaire*; 2. Bowen has been a catalog model; 3. Two of these three statements are false.

Table 10 You're at the build-a-necklace necklace-a-roo (a highly competitive event in Bark Silly, μtah). The necklaces look like this:

This competition seems highly saucy! Bring in Patricia!



You want to predict all of your opponent's designs (uniquely, up to re-orientations). Competitors can only choose from k colors for the beads, but the string lengths have to stay the same. How many unique necklaces do you, the necklace-a-roo master need to worry about?

Table 11 While looking for the next day's problem set in Darryl's room, Bowen finds Darryl's secret stash of triangle-shaped playing cards. He decides he should try a perfect 3-shuffle. Check out `go.edc.org/3piles`.

a. Create a table for the number of perfect 3-shuffles needed to restore different deck sizes.

b. For what deck sizes are the number of perfect 2-shuffles needed the same as the number of 3-shuffles?

Triangle card, triangle card, triangle card hates rectangle card, they have a fight, triangle wins, triangle card.

What's a 4-shuffle like? It's not important. Triangle card.

Table 12 Darryl and Bowen travel to Mirror Land, where only reflections exist. However, upon arrival they soon astonish all of the inhabitants of this land by showing them how to use two consecutive reflections to produce rotations. Given a regular n-gon, figure out how to use any two of its n reflective symmetries to produce any of its n rotational symmetries.

I'm looking at the man in the mirror. He's claiming that we can't rotate... Just take a look at yourself and make that change.

Mirror mirror on the wall, who is the reflectiest of them all?

Oh where oh where has Poly gone? *Poly went to get a cracker. Poly went to Nesia. Poly runs the gamUT.*

More 3-D Stuff

11. Kristen and Jake highly encourage you to go look at those problems involving the cube if you haven't already done them. They're fun! Start at Problem 12 from Day 11 then do Problem 8 from Day 12.

12. PCMI's logo is a regular octahedron. Enumerate the 24 rotational symmetries of a regular octahedron.

But you didn't have to come to Utah
Meet some friends and shuffle cards and then mod some numbers
I guess you've got to leave us though
Now we're just some math camp that you used to know

13. **a.** Jessica wants to color each vertex of a regular octahedron from a palette of k colors. How many colorings are unique up to rotations? Why does this answer look so familiar?

b. Max wants to color each face of a regular octahedron from a palette of k colors. How many colorings are unique up to rotations?

c. Anne wants to color each edge of a regular octahedron from a palette of k colors.

Schtuff and Schtuff

5. Draw a table.

4. Dylan insists in trying to do perfect shuffles on a deck of 52 cards, but splitting it into 10 piles. Define how these shuffles are going to work and figure stuff out.

Dylan also insists that he does not like your girlfriend. No way, no way.

3. Here are six functions.

- $m(x) = 1 - \dfrac{1}{1-x}$
- $o(x) = 1 - x$
- $n(x) = 1 - \dfrac{1}{x}$
- $i(x) = x$
- $c(x) = \dfrac{1}{x}$
- $a(x) = \dfrac{1}{1-x}$

a. Build an operation table for working with these six functions, where the operation is "composition". For example, $n(o(x)) = m(x)$.

b. Which, if any, other tables can match this table by pairing the entries in some way?

Now and then I think of all the times you gave me Neat Stuff
But had me believing it was always something I could do
Yeah I wanna live that way
Reading the dumb jokes you'd play
But now you've got to let us go
And we're leaving from a math camp that you used to know

2. The number of orientations of a d12 and a d20 is the same. Are these isomorphic doyathink?

1. Figure out how to use shuffling to find the *base-10* decimal expansion of $\frac{1}{51}$ along with other fractions.

Don't you forget about us
We'll be alone, shufflin', you know it baby
Bracelets, we'll take them apart
Then put 'em back together in parts, baby
I say $(LA)^{55}$
When you walk on by
Will you call me maybe . . .
(See you again soon.)

No More Stuff