The Nature and Role of Reasoning and Proof

Reasoning and proof are fundamental to mathematics and are deeply embedded in different fields of mathematics. Understanding of both is needed for all.

Across the world in many countries however, reasoning and proof are being squeezed out of the curriculum for a variety of reasons, among them writing assessment items to evaluate proofs, time for taking a test with proof items, the cost of grading those items, and time used in classrooms to prepare for tests. Because reasoning and proof are so integral to mathematically literate adults, the mathematical community must find ways to address these barriers.

In this brief, reasoning and proof are defined and illustrated, and an argument made about why these should not be left out of the mathematics curriculum. Brief discussions of understanding and constructing proofs and development of understanding of reasoning and proof along with examples that could be used at different developmental levels are provided in an appendix. [Note that ways of describing developmental levels may vary from country to country.]

What is reasoning?

Reasoning consists of all the connections, between experiences and knowledge that a person uses to explain what they see, think and conclude. Reasoning and proof permeate our world, even in children's literature. Alice, from the logician/mathematician Lewis Carroll's *Alice in Wonderland*, said, "I say what I mean or at least I mean what I say. They're the same thing you know." Several other characters in the book take her to task for those statements. Young children should understand that Alice's reasoning is faulty perhaps by thinking about comparable sentences such as, "I eat what I like or at least I like what I eat. They're the same thing you know."

Mathematically literate adults use reasoning to synthesize or decide on the validity of claims or "proofs" in their daily lives. For example, knowing that a survey reports 9 of 10 people interviewed believe that *Avoidapain* is a good medicine to relieve headaches, should not automatically be a basis for drawing conclusions about the effectiveness of *Avoidapain*. Mathematically literate adults should recognize that the given information does not indicate whether the participants were randomly chosen from a suitable population, that an adequate number of people were interviewed to make any conclusions, and that bias was eliminated in the way the survey was conducted. Without further information, conclusions can only be drawn about the 10 people interviewed and not the population in general.

Reasoning is endemic to mathematics and is used by teachers and textbooks to explain why mathematical conclusions are correct, rather than appealing simply to authority. When students are asked to show their work or to justify their answers to mathematical questions, some form of reasoning is involved. This reasoning should be emphasized and made visible in both classrooms and on assessments.

Deductive reasoning involves chains of statements that are logically connected; this kind of reasoning, developed by the ancient Greeks, characterized both early mathematical thinking and logical thinking in other domains. While deductive reasoning is of special importance to mathematics, and is generally included in mathematical proofs, *inductive* reasoning is also important. Students use inductive reasoning when they look to generalize results or observations from a few cases. This kind of reasoning is used by young children in work with patterns of various kinds, identifying a pattern and checking whether it holds for a limited number of other cases. The same kind of reasoning is at the heart of the thinking older students and professional mathematicians use when engaged in experimental mathematical activity.

Student understanding of reasoning and proof develops slowly over school life. Thus, the curriculum and instruction need to pay appropriate attention to the development of pupil understanding. Reasoning and proof should not be restricted to particular courses and particular ages, but rather attention and emphasis should change over time. Young students should develop mathematical reasoning skills as a foundation for later and more sophisticated study.

Students should be introduced to forms of proof early, arguing from number line models or through the use of diagrams to explain an operation. As students progress into secondary school, the curriculum should be designed to engage them with particular forms of proof such as proof by contradiction, existence proof and proof by mathematical induction Systematic attempts to identify the likely progression of competence of students have been made by various education authorities (Western Australia Curriculum Framework, 2007) and curricular evaluations (Ubuz, 2007). Such examples allow teachers to recognize likely stages of development and allow them to adjust the experiences provided as well as the expectations of students accordingly.

What is proof?

In the everyday world, proof means many things to many people. A drug company may prove that a drug has no serious side effects by testing it on many people. A prosecutor may prove that a criminal is guilty ‘beyond all reasonable doubt’, using evidence and persuasion. In some countries, the phrase, “The proof of the pudding is in the eating” suggests that the quality of something can be determined by testing it in the real world. A marketing manager in a business may provide a ‘proof of concept’ to gain approval of the Board of Directors to promote a product. In each case, a conclusion is reached on the basis of evidence, with the intention of persuading an audience that something is true.

In mathematics, the meaning of proof is unlike these everyday meanings. Here, reasoning, usually deductive, is central in proving mathematical claims. A mathematical proof comprises a logical argument with carefully stated assumptions, statements using precise language and definitions, and reasoning used to reach a valid conclusion. It does not depend on gathering data (as in the case of drug testing) on personal persuasion (as in the law court), or on the voice of authority (as for the Board of Directors) but relies on the logical argument alone. The nature of a proof depends on the mathematical sophistication of the prover. Whether or not the proof is accepted may also depend on the mathematical background of the audience.

There are many different types of mathematical proof. In school mathematics common proof types are direct proof, proof by exhaustion, proof by contradiction, existence proof and proof by mathematical induction. Examples of these different proof types follow.

Direct proof

A direct proof has a chain of statements, each of which follows logically from the previous one. To illustrate such a proof, consider a mental arithmetic procedure to write down the square of a two-digit number whose units digit is five. The square of the number (such as 65) has as leading digits (6, in this case) multiplied by one more then the tens digit (6 + 1 =7, in this case) followed by 25. That is, $65^2 = 4225$. To prove that this algorithm works for all two-digit numbers with 5 as the units digit, consider the following:

> Represent the number as $10a + 5$, with the tens digit represented by a where a is a natural number.
>
> Then the square of the number is
>
> $$\begin{aligned}(10a + 5)(10a + 5) &= 100a^2 + 50a + 50a + 25 \\ &= 100a^2 + 100a + 25 \\ &= 100(a^2 + a) + 25 \\ &= 100[a(a + 1)] + 25\end{aligned}$$
>
> The resulting product has $a(a + 1)$ forming the leading digits, followed by 25, as required.

Note that in this case, a student writing or reading the proof needs to be familiar with the distributive law for multiplication over addition for natural numbers and the decimal place value system. Although the proof rests on these assumptions, these mathematical properties are not stated explicitly as they are accepted as true.

Proof by exhaustion

A proof by exhaustion relies on checking all cases. In some fields of mathematics, such as number theory, it is a common practice to use categories that exhaust all possible cases to construct a proof.

To illustrate proof by exhaustion, a computer or calculator program or a spreadsheet may be written to prove that there are only two three-digit numbers with the property that the numbers themselves are the sum of the cubes of their digits:

$153 = 1^3 + 5^3 + 3^3$
$407 = 4^3 + 0^3 + 7^3$

In this case, the proof depends on the completeness of the program or the spreadsheet, checking each number from 100 to 999 inclusive. While some readers may accept the result, based on their interpretation of the program or spreadsheet, others may demand to see the complete list to be satisfied that it is complete.

Proof by contradiction

Proof by contradiction relies on understanding the relevant aspects of logic. This idea underpins the important concept of statistical hypothesis testing. A famous example of proof by contradiction is the proof that there are infinitely many prime numbers.

> Assume that there is a finite number, k, of ordered prime numbers, represented symbolically as $p_1, p_2, \ldots, p_k$ where p_k is the greatest.
> Multiply all the prime numbers together to obtain the product P.
> $P = p_1 \times p_2 \times p_3 \times \ldots \times p_k$
> Now consider the number $P + 1$. Because prime numbers are positive, the product of all the primes, P, must be greater than any individual prime number. And $P + 1$ is greater than P. Now P is a number greater than any of the prime numbers and cannot be one of them.
> When $P + 1$ is divided by any of the prime numbers, $p_1, p_2, \ldots, p_k$, there is a remainder of 1.
> So $P + 1$ must itself be a prime number (since it has no factors other than 1 and itself).
> This new prime number $P + 1$ is greater than any of the prime numbers, which is a contradiction of the claim that p_k is the greatest prime number.
> Therefore the original assumption, that there are a finite number of prime numbers, must be false.
> So the number of prime numbers must be infinite.

This proof was known and celebrated thousands of years ago as an example of the power of mathematical reasoning.

Existence Proof

An existence proof determines that a particular object actually exists, although the proof may not produce the object in question. For example, consider the problem of constructing a square in a triangle, with all four vertices of the square on the triangle.

> Figure 1 shows how a rectangle *ABCD* can be constructed inside the triangle, with *AB* smaller than *BC*.
>
> Figure 1: Triangle with Rectangle *ABCD*; *AB* < *BC*

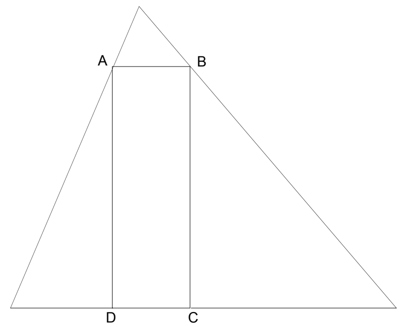
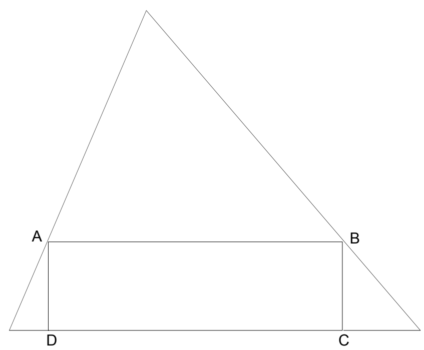


Figure 2 shows how rectangle *ABCD* can be constructed inside the triangle with *AB* greater than *BC*.

Figure 2: Triangle with Rectangle *ABCD*; *AB* > *BC*



Now, imagine starting with Figure 1 and gradually changing the rectangle by moving line segment *AB* downwards towards the horizontal base of the triangle. At some point before Figure 2 is reached, the rectangle *ABCD* will be a square, with *AB* = *BC*.

Notice in this case that the proof establishes that the desired square exists, but it does not construct the square.

Proof by mathematical induction

Figure 3 shows a geometrical argument why the sum of the consecutive odd natural numbers might be a perfect square. The drawing suggests the following:

1

1 + 3 = 4

1 + 3 + 5 = 9

1 + 3 + 5 + 7 = 16

1 + 3 + 5 + 7 + 9 = 25

Figure 3: Geometrical depiction of sum of consecutive odd natural numbers

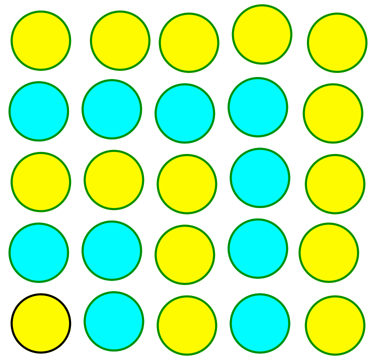


Figure 3 provides a visual "proof" that the sum of the first n odd natural numbers might be n^2. Though Figure 5 produces a good visual argument, the visual argument in this case is not a proof that the statement "The sum of the first n odd natural number is n^2.

A form of proof often associated with series is proof by mathematical induction. Usually, students would not encounter proofs of this kind until late in secondary school. To illustrate, consider the claim that the sum of the first n odd natural numbers, represented by S_n, is n^2.

To Prove: $S_n = 1 + 3 + 5 + \ldots + (2n - 1) = n^2$.
Note that the nth odd number is $2n - 1$ and the next consecutive odd number is $2n + 1$.
We noted from Figure 5 that the claim is true for
$S_1 = 1 = 1^2$
$S_2 = 1 + 3 = 4 = 2^2, \ldots,$
$S_5 = 1 + 3 + 5 + 7 + 9 = 25 = 5^2$
It appears that a sum of consecutive odd natural numbers is added to the next odd natural number, the result is the square of that next off natural number. To examine whether this might be true in general, assume the claim is true for some particular value of k, so that the sum of the first k odd natural numbers is k^2:
i.e., assume $S_k = 1 + 3 + 5 + \ldots + (2k - 1) = k^2$.
If this "next case" can be built on the kth case and shown to be true, then the general statement is true for all natural numbers.
Here, we add the next odd number to the statement assumed to be true. That is, the $(k + 1)^{th}$ odd number, $(2k + 1)$ is added to S_k giving S_{k+1} . Thus,

$$S_{k+1} = S_k + 2k + 1 = 1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) = k^2 + (2k + 1)$$
$$\text{i.e., } S_{k+1} = k^2 + 2k + 1$$
$$\text{i.e., } S_{k+1} = (k + 1)^2$$

So, if the claim is true for any value of k, it is also true for $k + 1$.
It is true for $k = 1$, so it must be true for all values. So $S_n = n^2$.

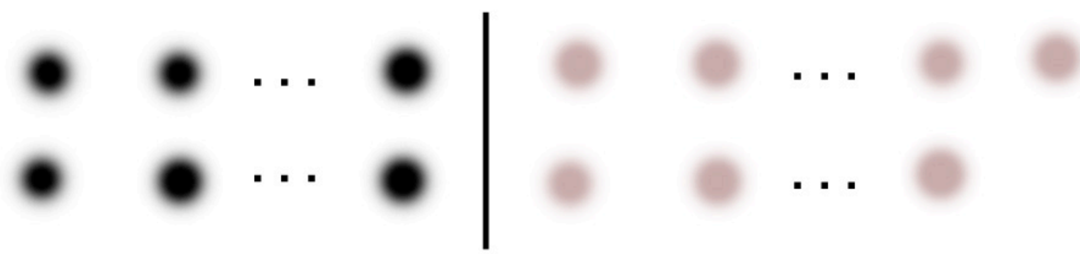
What form of proof?

While some theorems are best proved with one kind of proof, many theorems can be proven by many different proofs. An outstanding example is the Pythagorean Theorem, for which hundreds of different proofs have been constructed. It is helpful for students to encounter more than one proof, or kind of proof, for a theorem, to support the view that *a* proof rather than *the* proof is important. Indeed, useful classroom discussion can be generated regarding which of several proofs is preferred, to support the idea of developing an ‘elegant’ proof, with aesthetic overtones. For example, the proof of the infinite number of primes above, often attributed to the ancient Greek geometer Euclid, is still regarded by many mathematicians as elegant, perhaps because it shows the enormous power of mathematical reasoning in only a few lines.

Understanding a proof

An important part of reasoning and proof is understanding the proofs generated by other people. Students need to learn how to read proofs, understanding the various component steps and the logical relationships between them. Understanding a proof depends on the background and assumptions being made by the person doing the proof. For example, proof for young students may not be considered proofs by those with more experience, but young children may produce logically correct proofs. A young child might successfully prove that the sum of two consecutive natural numbers is odd with the following diagram and appropriate words. Figure 4 might show a child’s thinking; the language used might not be sophisticated, but the reasoning for a proof is seen.

Figure 4: Child’s drawing of an even number of dots plus an odd number of dots

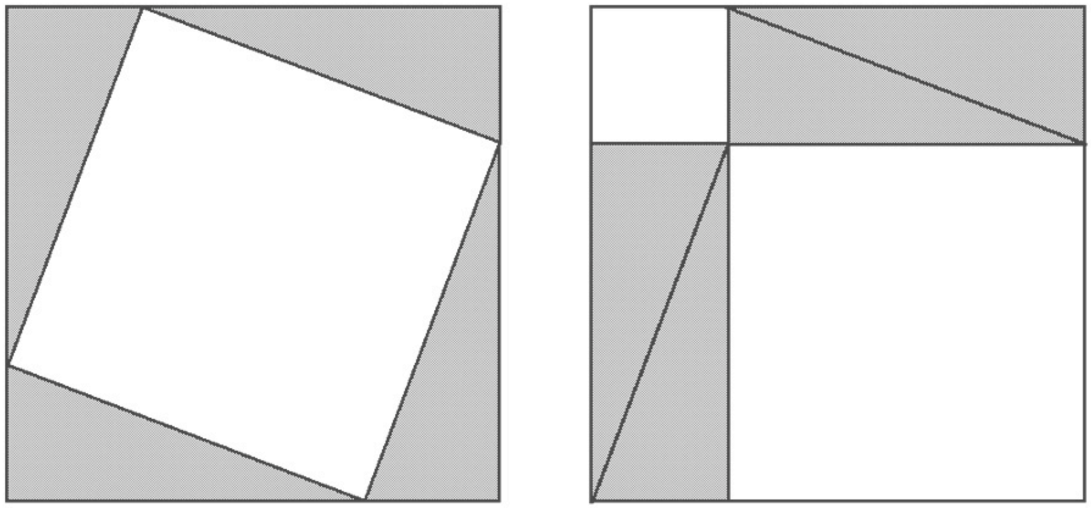


In Figure 4 to the left of the vertical bar, there is an even number of circles. To the right of the bar are an odd number of circles. The sum of all the circles must be an odd number because one circle has no match. Teachers should consider this a good proof from a child because the reasoning is logical and it could be extended for any two consecutive natural numbers.

A more sophisticated version of the argument might be as follows: One of these numbers must be even, $2a$, [a is a natural number] and the other must odd and consecutive, $2a + 1$. The sum is $4a + 1$ which may be written as $2(2a) + 1$, an odd number.

Valid proofs of the Pythagorean Theorem that in a right triangle with legs of length, a, and b and hypotenuse c, $a^2 + b^2 = c^2$ have appeared in many books. Frequently there are no statements or reasons given as seen in Figure 5, but areas of the figure on the left can be compared to that of the figure on the right to complete the proof.

Figure 5: “Proof” of the Pythagorean Theorem



When one constructs a proof, it is important to understand that once something is proved, there are no counterexamples that contradict the proof. A proof is true in all circumstances under the conditions by which it was constructed. Thus, not only does understanding a proof constitute being able to recognize what is and what is not a proof, it also must include recognizing that a proof means that there are no exceptions from the proof.

An example of a conjecture that had not been proved for many years was one from Leonhard Euler who conjectured that there were no integers, *x*, *y*, *z*, and *w* such that $x^4 + y^4 + z^4 = w^4$. It was almost 200 years later that Naom Elkes from Harvard University showed that $2{,}682{,}440^4 + 15{,}365{,}639^4 + 18{,}796{,}760^4 = 20{,}615{,}673^4$. This disproved the conjecture. Had the conjecture been true, no counterexample such as this could have been found (de Villiers, 2004).

For those who may study mathematical sciences, it is expected that they will eventually be able to understand proofs in many different forms. Some examples might include mathematical induction and understanding when a "proof" by mathematical induction might not be true. A classic example is the "proof" that all horses are the same color as seen below:

> If there is only one horse, then it is the same color as itself.
> Assume that k horses are the same color. Now consider $k + 1$ horses. Remove 1 horse and there are k horses left. They are the same color by the assumption. Now put the $k + 1^{st}$ horse in the group and remove a different horse. The k horses left are the same color so the entire group of $k + 1$ horses are the same color.

Students and teachers should recognize that the proof is not true. What makes it not true is that the conditions for the use of mathematical induction are not met.

As the examples illustrate, an understanding of proof is essential for all, but the level of understanding may vary.

Constructing a proof

The *constructing* of simple or more complicated proofs can and should occur at almost all levels of schooling with almost all students. The degree of rigor in the proofs depends on the level of both prover and audience. Students who study the mathematical sciences beyond secondary school will need a stronger background in construction of proofs than those who do not, but the idea of proof is so central to mathematical thinking that no students ought be deprived of it on the grounds that they have non-mathematical career trajectories. In general, it is more difficult for students to construct proofs than it is to understand proofs constructed by other people.

An important aspect of proving is making the reasoning visible. Depending on the level on which the proving takes place, the outcome can be a drawing, verbal language or symbolic language. Proofs usually contain a logic chain. Explicating the logic is not an algorithmic process.

A self-made conjecture can be based on experiments. These experiments can be done with the use of dynamic geometry environment, graphing calculator, computer algebra system software or simply paper and pencil. Before attempting to construct a proof, it is desirable to see if a counterexample to disprove the conjecture may be found. After thoroughly testing the conjecture, or other ways to get convinced that the conjecture seems to be true, the real constructing of the proof may begin.

Proofs in published materials, such as those in school textbooks, have generally been carefully edited to remove all unnecessary steps and to improve the flow of the argument. Students and teachers ought to expect that first attempts to prove something will generally be less carefully constructed than this, and that a process of refinement may be needed to reach the same standards.

Motivation for proof

Within mathematics, conjectures continue to be regarded as such, until a proof is provided. This encourages people to seek proofs of conjectures, to put the status of their truth beyond doubt. Famous recent examples of this are the four-color theorem (that no more than four colors are needed to color any planar map) and Fermat's Last Theorem (that there are no integers x, y, z and n, with n, greater than 2 satisfying $x^n + y^n = z^n$), both of which were proved late in the twentieth century, after many years of attempts by professional mathematicians as well as amateurs. There continue to be some unresolved conjectures in mathematics, such as the conjecture that there are an infinite number of twin primes (prime numbers that differ by 2, such as 11 and 13, 17 and 19) or Goldbach's conjecture that every positive integer can be expressed as the sum of two prime numbers.

While professional mathematicians might be motivated to search for proofs of conjectures that have baffled others for many years, it is unlikely that this will be sufficient or productive for school students. Instead, care needs to be taken to find ways to motivate students to search for proofs, to help them to understand the importance of such activity within their developing competence in mathematics. Students may not see the importance of proving statements that seem obvious to them, or that seem to arise without a supporting context, such as a series of explorations or examples.

The most powerful ways of motivating proof involve providing students with an environment to make conjectures by themselves and encouragement to systematically explore them, leading to a proof to remove all doubt. Some students are motivated when provided with situations where they have to predict and then determine whether their predictions are valid. The use of such open-ended situations and mathematical investigations are good ways of initiating such work, allowing students to engage in a form of mathematical experimentation. Opportunities to work within a small group will increase the likelihood of students being motivated to prove that their own conjectures are correct, in order to persuade fellow students or classmates.

Computers and calculators have the potential to motivate proof, as they provide new opportunities to experiment with mathematical ideas and objects, to detect patterns and regularities, leading to conjectures that require proof. Many teachers have reported on ways of using dynamic geometry systems, spreadsheets, calculators and other mathematical tools for such purposes.

Conclusion

In this brief, the critical roles of reasoning and proof in mathematics have been described and illustrated. A mathematics curriculum that does not support the development of understanding of these is not providing students with access to the key distinguishing feature of the discipline. While the age and circumstances of students need to be taken into account appropriately, it is unacceptable for students to be denied appropriate opportunity and support to learn about reasoning and proof in mathematics over the course of their schooling.

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