## Problem Set 1: Kite Club

Welcome to the course! We know you'll learn a lot of mathematics here—maybe some new tricks, maybe some new perspectives on things with which you're already familiar. A few things you should know about how the class is organized:

- **Don't worry about completing all the questions.** If you're completing every question, we haven't written the problem sets correctly.
- **Don't worry about getting to a certain problem number.** Some participants have been known to spend the entire session working on one problem (and perhaps a few of its extensions or consequences).
- **Stop and smell the roses.** Getting the correct answer to a question is not a be-all and end-all in this course. How does the question relate to others you've encountered? How do others think about this question?
- **Be excellent to each other.** Believe that you have something to learn from everyone else. Remember that everyone works at a different pace. Give everyone opportunity to express themselves.
- **Teach only if you have to.** You may feel tempted to teach others in your group. Fight it! We don't mean you should ignore people, but don't step on someone else's A-HA! moment—give everyone the chance to discover. If you think it's a good time to teach your colleagues about Voronoi diagrams, think again: The problems should lead to appropriate mathematics rather than requiring it.
- **Each day has its Stuff.** There are problem categories: Important Stuff, Neat Stuff, Tough Stuff. Check out the Opener and Important Stuff first. All the mathematics that is central to the course can be found and developed in the Important Stuff. *That's* why it's Important Stuff. Everything else is just neat or tough.

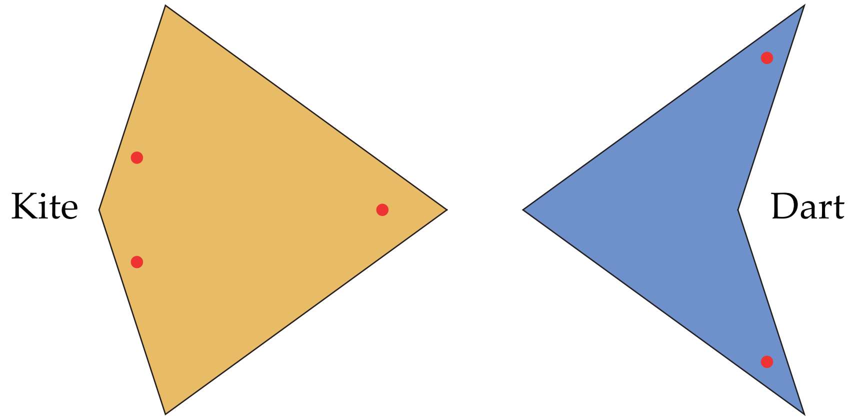
Some problems are actually *unsolved*. Participants in PCMI courses like this have settled at least two unsolved problems.

When you get to Problem Set 2, go back and read the introduction again.

Don't worry, you'll get there!

## Opener

**1.** Cut out the polygons on the handout. The larger polygon is called a *kite* and the smaller polygon is called a *dart*.




For the next 12 minutes, use the dotted kites and darts to create tilings (arrangements with no holes or overlapping pieces). But, you must adhere to two rules.

This activity is more difficult in New England where "dart" and "dot" are said the same way.

**Rule #1:** Vertices can only touch other vertices, not the middle of edges. If two pieces share an edge, they must share the entire edge.

**Rule #2:** Edges must line up so that their "dots" are together.

Rule #3: *You do not talk about Kite Club.*

Start a new tiling whenever you want. Write down your observations, especially any "recurring themes" you notice while tiling.

*When 12 minutes are up, move on to Problem 2.*

*Tick tock . . .*

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### Important Stuff

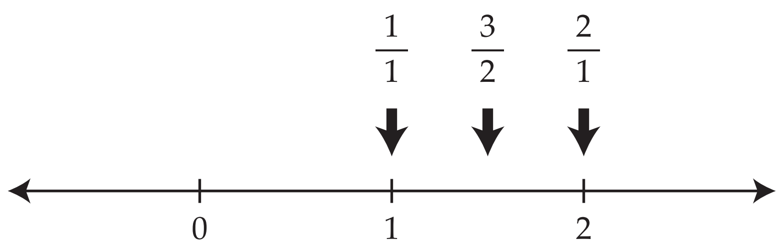
**2.** There are only 7 different ways that a vertex can be surrounded with kites and/or darts, while adhering to all of the rules. Find them! What is true of all 7 ways?

**3.** Get a piece of $8\frac{1}{2}$-by-$5\frac{1}{2}$ paper. (Fold letter-size paper in half). Cover the paper with kites and darts (following all of the rules) until you can't see the paper any more. Count how many kites and darts are touching the paper. Gather some data from around the room and make conjectures about the ratio between kites and darts in a tiling.

**4.** Start with 0 and 1, then keep adding two terms to get the next:

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots$$

Use consecutive Fibonacci numbers (value/previous value) to make fractions and place these fractions on this number line. Keep placing! What do you notice?

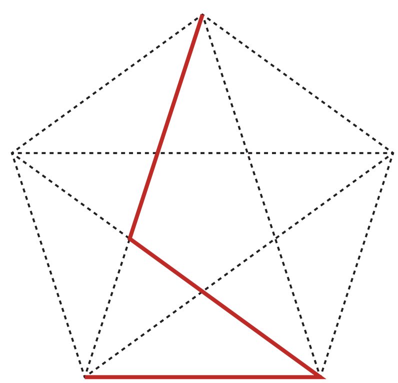



Eventually these numbers start multiplying like rabbits. Wait, no, they're adding like rabbits?

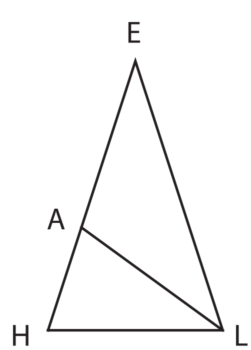
### Neat Stuff

**5.** Jason measured these three marked segments in a regular star pentagram. He couldn't decide which was longest. Which one is longest? Explain how you know.

It's OK to skip around among the Neat and Tough Stuff problems. As they say in some counting problems, "order doesn't matter" here.



**6.** Here's the center triangle from Problem 5.




**a.** Which triangle is a scaled copy of which other triangle? Be as precise as you can.

**b.** Give a brief explanation of *why* these triangles are scaled copies.

Two objects are *similar* if they are scaled copies. Congruent objects are similar with scale factor 1.

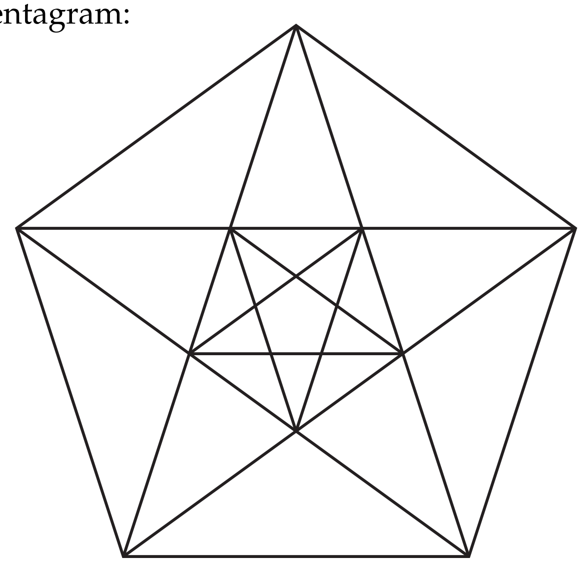
**c.** Replace these question marks in a way that must form a true proportion:

$$\frac{EH}{??} = \frac{??}{??}$$

**d.** If $LH = 1$, estimate the length of $EH$ to two decimal places.

Wait, which length is that? $EH$.

**7.** Lindsay says it's cool to put a star pentagram *inside* a star pentagram:



The kite and dart polygons are in Lindsay's diagram! Find them. Use the diagram and/or your work from Problem 2 to find the measures of all angles in the kite and dart.

**8.** The golden ratio $\phi$ can be defined several ways. One way is that it's a positive number so that $\phi^2 = \phi + 1$.

This letter is pronounced like either "fee" or "fie."

**a.** Using the definition above, without using or figuring out the value of $\phi$, show that $\phi^3 = \phi(\phi + 1)$.

But $\phi^3$ is not pronounced "foe."

**b.** Show that $\phi^3 = \text{blah}\phi + \text{bleh}$. You figure out the blahnks, but they're integers.

**c.** Show that $\phi^4 = \text{blih}\phi + \text{blöh}$, again without evaluating $\phi$.

And no, $\phi^4$ is not pronounced "fum."

**d.** Show that $\phi^5 = \text{bluh}\phi + \text{blyh}$.

**e.** Describe a general rule for $\phi^n$. Awesome!!

**9.** Kim wants to build a huge kite-and-dart tiling, extending in every direction forever. Estimate the ratio of the number of kites to darts Kim would need.

**10.** If you didn't have to follow the matching rule, what are all possible ways to surround a vertex with kites and darts?

**11.** Suppose the shortest side of each kite and dart has unit length. For both the kite and dart, find all side lengths, the area, and the length of the two diagonals.

**12.** In Problem 7, Lindsay drew a pentagram inside a pentagram. The larger pentagram is how many times larger (in area) than the smaller pentagram? Give your answer in terms of integers and Greek letters only, please.

## Tough Stuff

**13.** What is the ratio of karts to dites in an infinite kite-and-dart tiling? Prove it!

# Problem Set 2: Tiles To Go Before I Sleep

## Opener

**1.** Look for the handouts with the big star and big sun. Use your kites and darts from Set 1 to tile the big star and big sun, adhering to the rules. On the big sun, one piece has already been placed for you. Complete the rest of the tiling.

It's Set 2. Did you remember?

The Big Star and Big Sun are not college football conferences. Yet.

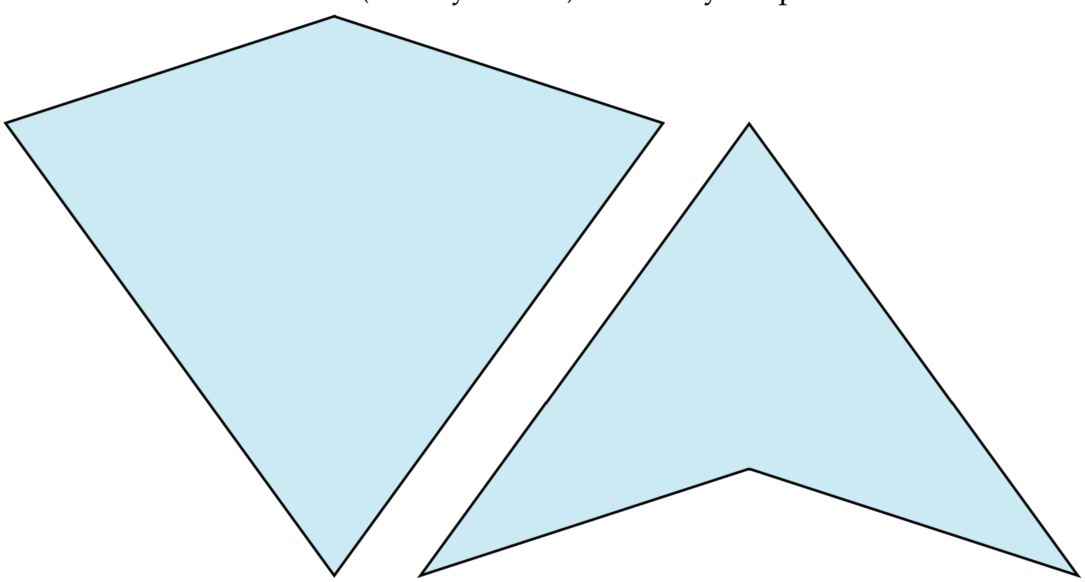
Fill in these blanks:

1 big star = ______ kites and ______ darts

1 big sun = ______ kites and ______ darts

### Important Stuff

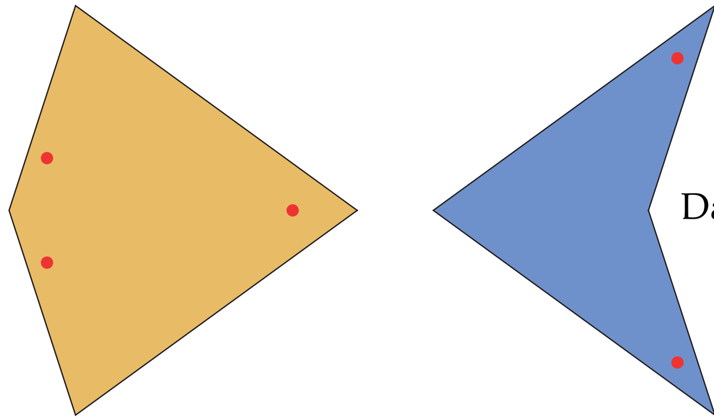
**2.** Kayleigh notices that we used five big kites to make the big sun. Look at the pieces that you used to cover a big kite. Place those pieces onto the big kite below. You will have to cut (or maybe fold) some of your pieces.



Similarly, the big star was created using five big darts. Place the pieces that covered a big dart onto the big dart above.

Big suns and big stars, sounds like some sort of theory. Quick, call Sheldon.

**3.** Pictured below are one kite and dart, regular size. Draw in a miniature version of your work from Problem 2 on the shapes below.




**4.** Fill in these blanks.

If you had to cut/fold some pieces, what happens to the area?

1 regular kite's area = area of ______ mini-kites
and area of ______ mini-darts
1 regular dart's area = area of ______ mini-kites
and area of ______ mini-darts

**5.** Create your own tiling with your cut-out pieces, using 8 or fewer pieces. Now turn each piece over carefully in place and draw in the lines you drew for Problem 3. Voilà! You just created another tiling involving mini-kites and mini-darts.

It's fun to say "Voilá!"

**a.** How many kites and darts did your original tiling have? That same area is now covered by the areas of mini-kites and mini-darts. Count them! Remember to count half-shapes as half-areas.

I shall call it . . . Mini-$\phi$.

**b.** How could you use the information from Problem 4 to reach the same conclusion?

**c.** If within each mini-kite and mini-dart, you drew in mini-mini-kites and mini-mini-darts, how many mini-mini-kites and mini-mini-darts would you have? Try to do this without counting them.

**d.** Jennifer yells "Mini-mini-mini!"

That's the name of the next-generation iPad.

**e.** Calculate the ratio of kites to darts in each generation of your work.

**6.** Jessica decides to start Problem 5 as simply as possible: one kite.

One is the loneliest number of kites that you'll ever do.

- **a.** Replace the kite with mini-kites and mini-darts. How many mini-kites and mini-darts are there? Count two halves as one . . . sum of two halves.

Two kites can be as bad as one . . .

- **b.** Now minify your mini-arrangement again. How many mini-mini-kites and mini-mini-darts will you have?
- **c.** Yes... Of course. Did you need to ask our permission to keep going?
- **d.** What do you notice? No way.

"Can I keep going?"
"I don't know, *can* you?"

**7.** Cesar decides to start Problem 5 with one dart instead of one kite. Follow the process from Problem 6 and see what happens when the dart is mini-mini'd.

Eenie, mini-mini-moe.

**8.** Determine the ratio of the number of kites to darts in an infinite kite-and-dart tiling.

Way!

## Neat Stuff

**9.** Here's something interesting about $\phi^{-10}$:

$$\phi^{-10} = a \cdot \phi + b$$

where $a$ and $b$ are integers. Find $a$ and $b$ by yourself, then verify the result with technology. What does this tell you about the value of $\phi$?

**10.** Set 1 asked you to look at the ratios of consecutive Fibonacci numbers. What happens when you begin with a different starting pair? For example, the *Lucas numbers* follow the same rule as the Fibonacci numbers, but start with 2 and 1:

The Lucas numbers live upstairs from the Fibonacci numbers, on the second floor.

$$2, 1, 3, 4, \ldots$$

What happens to the ratio of consecutive Lucas numbers? Try some other starting pairs and see what develops.

**11.** Here's an interesting function.

The output of a function is determined by what you put in. For example,

$$f(42) = 1 + \frac{1}{42} = \frac{43}{42}.$$

$$f(x) = 1 + \frac{1}{x}$$

- **a.** What's $f(1)$?

**b.** Find the exact values of $f(2)$ and $f(\frac{3}{2})$.
**c.** Marianthi starts with $x = 1$, then she runs the function repeatedly on outputs. What happens?
**d.** Chase starts with $x = \frac{1}{2}$ then does what Marianthi did. Notice anything?
**e.** Escher starts with $x = 0$. What happens?

You might notice more here by using exact values instead of decimals.

**12.** If you drop the dot-matching rule for kites and darts, what other kinds of tilings can you get? Make some, and figure out the ratio of kites to darts in those tilings.

**13.** **a.** When you created the big sun and big star, why didn't you have to cut any pieces in half?
**b.** The kites and darts only have edges of two different lengths. How many different, valid, ways can you line up any two pieces along a long edge?
**c.** For each possible way, draw mini-kites and mini-darts inside of your pieces. What do you notice?
**d.** Use your diagrams to explain why, for any *valid* kite and dart tiling, replacing every kite and dart with miniature versions will not require any cutting of pieces, except possibly along the exterior edge of your tiling.

What does this say about infinite tilings of the plane?

**14.** Prove that any valid tiling will still be valid after kites and darts are replaced with mini-kites and mini-darts.

### Tough Stuff

**15.** In Problem 10 you noticed something. Find a nonzero starting pair for which it does *not* happen that the ratio of consecutive terms never approaches $\phi$. Yes, they do exist.

And yes, they're nonzero!

### More Stuff

These problems are from *Fractions, Tilings, and Geometry* by Kerins, Yong, Cuoco, Stevens, and Pilgrim, published in 2017 by the American Mathematical Society, available through your favorite online bookstores. The book was based on the PCMI morning mathematics course from 2014.