## Problem Set 1: Cut-Up

Welcome to the course! We know you'll learn a lot of mathematics here-maybe some new tricks, maybe some new perspectives on things with which you're already familiar. A few things you should know about how the class is organized:

- Don't worry about completing all the questions. If you're completing every question, we haven't written the problem sets correctly.
- Don't worry about getting to a certain problem number. Some participants have been known to spend the entire session working on one problem (and perhaps a few of its extensions or consequences).
- Stop and smell the roses. Getting the correct answer to a question is not a be-all and end-all in this course. How does the question relate to others you've encountered? How do others think about this question?
- Be excellent to each other. Believe that you have something to learn from everyone else. Remember that everyone works at a different pace. Give everyone opportunity to express themselves.
- Teach only if you have to. You may feel tempted to teach others in your group. Fight it! We don't mean you should ignore people, but don't step on someone else's A-HA! moment-give everyone the chance to discover. If you think it's a good time to teach your colleagues about Voronoi diagrams, think again: The problems should lead to appropriate mathematics rather than requiring it.
- Each day has its Stuff. There are problem categories: Important Stuff, Neat Stuff, Tough Stuff. Check out the Opener and Important Stuff first. All the mathematics that is central to the course can be found and developed in the Important Stuff. That's why it's Important Stuff. Everything else is just neat or tough.

When you get to Problem Set 3, go back and read the introduction again.

Some problems are actually unsolved. Participants in PCMI courses like this have settled at least two unsolved problems.

## Opener

1. Use the cut-out regular polygons provided to create arrangements where at least one vertex is completely surrounded and there are no "holes" or overlapping pieces. (The lengths of all sides are the same.) For each arrangement you produce, find the total number of vertices, edges, and polygon faces in the arrangement. For example, Kate created this arrangement with 6 vertices, 8 edges, and 3 polygon faces:


Kate's arrangement has no "holes" but also does not have a vertex that is completely surrounded by polygons. Do better than Kate by making arrangements that completely surround a vertex. Look for some patterns among your table's results.

## Important Stuff

2. What has to happen at a vertex for it to be completely surrounded? Give some examples of how this can happen, and an explanation of how it might not happen.
3. Imagine an arrangement that covers the entire plane, made from copies of only one regular polygon. Which regular polygons make this possible?
4. Explain why there can't be an arrangement of regular pentagons that completely surrounds a vertex without overlapping.

A regular polygon has all side lengths equal, all angle measures equal, and takes Metamucil before bed.

Don't double-count vertices and edges, but do count every vertex and edge, even those that end up "inside."

Try counting Kate's arrangement and make sure you get 6 vertices, 8 edges, and 3 polygon faces.

Wait, I thought math problems weren't supposed to start with "Imagine" . . .
5. Tracia slices a regular pentagon into three cool triangles.

a. What can you say about the angles in triangle 1?
b. What is the sum total of all the interior angles in a regular pentagon?
c. Are triangles 1,2 , and 3 congruent? What kind of triangles are they?
d. Find the exact measure of every angle in triangles 1, 2 , and 3 .

## Neat Stuff

6. Here's a regular star pentagram. Find four pairs of triangles where one is a scaled copy of the other.

7. For each pair of triangles you found in Problem 6, how many times longer is each side of the larger triangle? Measure and estimate, to two decimal places.
8. Dayna decided that each side of the large pentagon had length 1 . She then found other segments in the star pentagram that she says have length 1. Do these exist? Find some or explain why they do not exist.
9. Agree with Dayna and suppose the large pentagon's side length is 1 . Find an equation involving $x$, the length of the longest possible diagonal in Problem 6.
10. If Tracia's pentagon from Problem 5 has side length 1 , find the exact length of every side in triangles 1,2 , and 3 .
11. Which of triangles 1,2 , and 3 from Problem 5 has the largest area? How many times larger is it than the others?
12. Kyle loves Tetris! Its pieces have four squares connected, and are sometimes called tetrominoes. Kyle wants to figure out which of the 7 Tetris pieces can tile an entire plane. But, per usual Tetris rules, the pieces can only be rotated when put in place. How many of the 7 Tetris pieces can tile the plane?

## Tough Stuff

13. Kyle also loves Tetris ${ }^{5}$ ! Its pieces have five squares instead of four, and are sometimes called pentominoes.
a. How many pentominoes are there?
b. Tetris ${ }^{5}$ has more than this many pieces. Why? How many pieces does it have?
c. How many pentominoes can tile an entire plane? Reflections are allowed.
d. Kyle wants to figure out which of the Tetris ${ }^{5}$ pieces can tile an entire plane, using only rotations. How many Tetris ${ }^{5}$ pieces can tile the plane?
14. Find the smallest Tetris-style shape that does not contain any holes and cannot tile the plane in any way, including reflection.

Warning: Tetris ${ }^{5}$ probably does not actually exist.

A Tetris-style shape can be drawn by shading squares on a sheet of graph paper.

## Problem Set 2: I Can See For Tiles And Tiles

## Opener

1. Check out this floor from a church in Seville, Spain:


Imagine the pattern continuing forever in all directions. Well, all directions besides "up" and "down."

## Important Stuff

2. Paul put a regular pentagon, hexagon and octagon together so they share a vertex. Does this work out? Explain why or why not.

3. Look at all the arrangements of regular polygons that you made on Set 1. For each arrangement, calculate $V-E+F$, where $V$ is the number of vertices, $E$ is the number of edges, and $F$ is the number of polygon faces in the arrangement.
4. Start with 0 and 1 , then keep adding two terms to get the next:

$$
0,1,1,2,3,5,8,13, \ldots
$$

Use consecutive Fibonacci numbers (value/previous value) to make fractions and place these fractions on this number line. Keep placing! What do you notice?

5. Jason measured these three marked segments in a regular star pentagram. He couldn't decide which was longest. Which one is longest? Explain how you know.


Include interior edges and vertices.

Eventually these numbers start multiplying like rabbits. Wait, no, they're adding like rabbits?
6. Here's the center triangle from Problem 5.

a. Which triangle is a scaled copy of which other triangle? Be as precise as you can.
b. Give a brief explanation of $w h y$ these triangles are scaled copies.
c. Replace these question marks in a way that must form a true proportion:

$$
\frac{\mathrm{EH}}{? ?}=\frac{? ?}{? ?}
$$

d. If $\mathrm{LH}=1$, estimate the length of EH to two decimal places.

## Neat Stuff

7. You can take any non-square rectangle and chop a square out of it! In the diagram below, MEGN is a square chopped out of rectangle AHEM.

a. Suppose $E H=2$ and $A H=1$. Is rectangle GHAN a scaled copy of the original rectangle?
b. Suppose $E H=3$ and $A H=2$. Is GHAN a scaled copy of AHEM this time?
c. What if $E H=5$ and $A H=3$ ? Well, shoot.

In Set 1, this was Triangle 2. Now, it's got a triangle, too!

Two objects are similar if they are scaled copies. Congruent objects are similar with scale factor 1.

Wait, which length is that? EH.

AHEM! May I have your attention please! This is not a door. It is also not a fridge with the freezer on the bottom.
d. If $E H=x$ and $A H=1$, write a proportion that would have to be true if GHAN is a scaled copy of AHEM.
e. If $A H=1$, estimate the length of EH to two decimal places.
8. The golden ratio $\phi$ can be defined several ways. One way is that it's a positive number so that $\phi^{2}=\phi+1$.
a. Using the definition above, without using or figuring out the value of $\phi$, show that $\phi^{3}=\phi(\phi+1)$.
b. Show that $\phi^{3}=\operatorname{blah} \phi+$ bleh. You figure out the blahnks, but they're integers.
c. Show that $\phi^{4}=\operatorname{blih} \phi+$ blöh, again without evaluating $\phi$.
d. Show that $\phi^{5}=$ bluh $\phi+$ blyh.
e. Describe a general rule for $\phi^{n}$. Awesome!!
9. There are some ways to fit 3 regular polygons together perfectly to surround a vertex. Find as many as you can and write them as ordered triples ( $a, b, c$ ) where $a \leqslant b \leqslant c$ are the number of sides in the three polygons.
10. For each triple you found in Problem 9, there's something interesting about the sum of the numbers, the product, or the sum of the reciprocals. One of those.
11. Use what you found in Problem 10 to make a complete list of all the ways that 3 regular polygons can fit together at a vertex.
12. About what proportion of that Seville floor is made up of hexagons? squares? triangles?

## Tough Stuff

13. Suppose a regular pentagon has side length 1 . How far is it from one vertex to the midpoint of the opposite side?
14. Find the three regular polygons that come the closest to fitting together at a vertex, but don't because they overlap. Then find the three regular polygons that come the closest to fitting together at a vertex, but don't because they leave a super-tiny gap.

What's $x$ ? EH.

This letter is pronounced like either "fee" or "fie."

But $\phi^{3}$ is not pronounced "foe."

And no, $\phi^{4}$ is not pronounced "fum."

This problem is deliberately vague. Sorry about that.

## Problem Set 3: Kite Club

## Opener

1. Cut out the polygons on the handout. The larger polygon is called a kite and the smaller polygon is called a dart.


For the next 12 minutes, use the dotted kites and darts to create tilings (arrangements with no holes or overlapping pieces). But, you must adhere to two rules.

Rule \#1: Vertices can only touch other vertices, not the middle of edges. If two pieces share an edge, they must share the entire edge.
Rule \#2: Edges must line up so that their "dots" are together.
Start a new tiling whenever you want. Write down your observations, especially any "recurring themes" you notice while tiling.
When 12 minutes are up, move on to Problem 2.

## Important Stuff

2. There are only 7 different ways that a vertex can be surrounded with kites and/or darts, while adhering to all of the rules. Find them! What is true of all 7 ways?
3. Get a piece of $8 \frac{1}{2}$-by- $5 \frac{1}{2}$ paper. (Fold letter-size paper in half). Cover the paper with kites and darts (following all of the rules) until you can't see the paper any more. Count how many kites and darts are touching the paper. Gather some data from around the room and make
conjectures about the ratio between kites and darts in a tiling.
4. Lindsay says it's cool to put a star pentagram inside a star pentagram:


The kite and dart polygons are in Lindsay's diagram! Find them. Use the diagram and/or your work from Problem 2 to find the measures of all angles in the kite and dart.
5. Draw a line connecting the word "kite" with the word "dart" on Problem 1, dividing the kite and dart in half. Where have we seen the resulting triangles before? What are their angle measures?

## Neat Stuff

6. Kim wants to build a huge kite-and-dart tiling, extending in every direction forever. Estimate the ratio of the number of kites to darts Kim would need.
7. For each shape, build a tessellation using copies of that shape, or explain why it can't be done.
a. A non-square rectangle
b. An otherwise unremarkable parallelogram
c. A rhombus

This is just a fancy word for tiling.

A rhombus is a quadrilateral where all four sides have the same length. The plural is rhombuseses.
f. A trapezoid
g. The Z-lookin' piece from Tetris
8. Set 2 asked you to look at the ratios of consecutive Fibonacci numbers. What happens when you begin with a different starting pair? For example, the Lucas numbers follow the same rule as the Fibonacci numbers, but start with 2 and 1 :

$$
2,1,3,4, \ldots
$$

What happens to the ratio of consecutive Lucas numbers? Try some other starting pairs and see what develops.
9. Here's an interesting function.

$$
f(x)=1+\frac{1}{x}
$$

a. What's $f(1)$ ?
b. Find the exact values of $f(2)$ and $f\left(\frac{3}{2}\right)$.
c. Marianthi starts with $x=1$, then she runs the function repeatedly on outputs. What happens?
d. Chase starts with $x=\frac{1}{2}$ then does what Marianthi did. Notice anything?
e. Escher starts with $x=0$. What happens?
10. Set 2 asked about the ways that three regular polygons could fit at a vertex. Find some ways to fit four regular polygons at a vertex, and look for any relationship among the quadruples ( $a, b, c, d$ ) giving the number of sides of the regular polygons used.
11. Find every possible way $4,5,6$, or 7 regular polygons can fit at a vertex.
12. Kites and darts aren't regular polygons, but they have known angle measures. Look for a way to use the style of Problem 10 on kites and darts. If you didn't have to follow the matching rule, what are all possible ways to surround a vertex with kites and darts?
13. Start with a 203-by-77 rectangle. Chomp off the largest possible square from this rectangle, then continue chomping squares. What happens? Try again starting with different rectangles and see what you come up with.

The Lucas numbers live upstairs from the Fibonacci numbers, on the second floor.

The output of a function is determined by what you put in. For example,

$$
f(42)=1+\frac{1}{42}=\frac{43}{42}
$$

You might notice more here by using exact values instead of decimals.

It's okay to walk back to any previous problem at any time.

Using a shoehorn, grease, or a hyperbolic plane do not count as possible ways.

The mathematical term for this is a gnom-nom.
14. Suppose the shortest side of each kite and dart has unit length. For both the kite and dart, find all side lengths, the area, and the length of the two diagonals.
15. In Problem 4, Lindsay draw a pentagram inside a pentagram. The larger pentagram is how many times larger (in area) than the smaller pentagram? Give your answer in terms of integers and Greek letters only, please.
16. Remember $\phi$ ? Here's something interesting about $\phi^{-10}$ :

$$
\phi^{-10}=a \cdot \phi+b
$$

where $a$ and $b$ are integers. Find $a$ and $b$ by yourself, then verify the result with technology. What does this tell you about the value of $\phi$ ?

## Tough Stuff

17. In Problem 8 you noticed something. Find a nonzero starting pair for which it does not happen that the ratio of consecutive terms never approaches $\phi$. Yes, they do exist.
18. For a lot of positive integers $N>1$, there is an integer solution to

$$
\frac{4}{\mathrm{~N}}=\frac{1}{\mathrm{a}}+\frac{1}{\mathrm{~b}}+\frac{1}{\mathrm{c}}
$$

a. Find solutions for $N=2$ through $N=10$.
b. Find the smallest N for which there is no solution, or prove that a solution exists for every N .
19. What is the ratio of karts to dites in an infinite kite-anddart tiling? Prove it!

## Problem Set 4: Tiles To Go Before I Sleep

## Opener

1. Look for the handouts with the big star and big sun. Use your kites and darts from Set 3 to tile the big star and big sun, adhering to the rules. On the big sun, one piece has already been placed for you. Complete the rest of the tiling.

The Big Star and Big Sun are not college football conferences. Yet.

Fill in these blanks:

$$
\begin{aligned}
& 1 \text { big star }=\ldots \text { kites and ___ darts } \\
& 1 \text { big sun }=\ldots \text { kites and ___ darts }
\end{aligned}
$$

## Important Stuff

2. Kayleigh notices that we used five big kites to make the big sun. Look at the pieces that you used to cover a big kite. Place those pieces onto the big kite below. You will have to cut (or maybe fold) some of your pieces.


Similarly, the big star was created using five big darts. Place the pieces that covered a big dart onto the big dart above.

Big suns and big stars, sounds like some sort of theory. Quick, call Sheldon.
3. Pictured below are one kite and dart, regular size. Draw in a miniature version of your work from Problem 2 on the shapes below.

4. Fill in these blanks.

$$
\begin{aligned}
& 1 \text { regular kite's area }=\text { area of ___ mini-kites } \\
& \quad \text { and area of ___ mini-darts } \\
& 1 \text { regular dart's area }=\text { area of } \quad \text { and area of mini-kites mini-darts }
\end{aligned}
$$

5. Create your own tiling with your cut-out pieces, using 8 or fewer pieces. Now turn each piece over carefully in place and draw in the lines you drew for Problem 3. Voilà! You just created another tiling involving mini-kites and mini-darts.
a. How many kites and darts did your original tiling have? That same area is now covered by the areas of mini-kites and mini-darts. Count them! Remember to count half-shapes as half-areas.
b. How could you use the information from Problem 4 to reach the same conclusion?
c. If within each mini-kite and mini-dart, you drew in mini-mini-kites and mini-mini-darts, how many mini-mini-kites and mini-mini-darts would you have? Try to do this without counting them.
d. Jennifer yells "Mini-mini-mini!"
e. Calculate the ratio of kites to darts in each generation of your work.

If you had to cut/fold some pieces, what happens to the area?

It’s fun to say "Voilá!"

I shall call it . . . Mini- $\phi$.

That's the name of the nextgeneration iPad.
6. Jessica decides to start Problem 5 as simply as possible: one kite.
a. Replace the kite with mini-kites and mini-darts. How many mini-kites and mini-darts are there? Count two halves as one . . . sum of two halves.
b. Now minify your mini-arrangement again. How many mini-mini-kites and mini-mini-darts will you have?
c. Yes... Of course. Did you need to ask our permission to keep going?
d. What do you notice? No way.
7. Cesar decides to start Problem 5 with one dart instead of one kite. Follow the process from Problem 6 and see what happens when the dart is mini-mini'd.
8. Determine the ratio of the number of kites to darts in an infinite kite-and-dart tiling.

## Neat Stuff

9. So the ratio of kites to darts is . . . wait, really? Look back at Set 2's opener. What are the implications about finding a repeating tile among the kites and darts?
10. For each shape, build a tessellation using copies of that shape, or explain why it can't be done.
a. An isosceles triangle
d. Any quadrilateral
b. Any triangle
e. Any pentagon
c. A kite
f. Home plate
11. Set 2 asked you to chop squares out of rectangles. Turns out you can also add squares to rectangles.
a. Start with a 1-by-1 square. At each opportunity, attach the largest possible square to form a larger rectangle. Write down the dimensions of each rectangle as you go. What happens?
b. Start over with a 3-by-1 rectangle. What happens?
c. Start over with a 16.3-by-1.8 rectangle. What happens in the long run?

One is the loneliest number of kites that you'll ever do.

Two kites can be as bad as one...
"Can I keep going?" "I don't know, can you?"

Eenie, mini-mini-moe.
12. It's possible for a box to have the same surface area and volume, at least numerically, even if their units are different.
a. Find three examples of boxes whose surface area and volume are numerically equal.
b. Find all possible dimensions of the box if each side length is an integer.
13. If you drop the dot-matching rule for kites and darts, what other kinds of tilings can you get? Make some, and figure out the ratio of kites to darts in those tilings.
14. Construct a kite and dart using one or more regular decagons.
15. a. When you created the big sun and big star, why didn't you have to cut any pieces in half?
b. The kites and darts only have edges of two different lengths. How many different, valid, ways can you line up any two pieces along a long edge?
c. For each possible way, draw mini-kites and minidarts inside of your pieces. What do you notice?
d. Use your diagrams to explain why, for any valid kite and dart tiling, replacing every kite and dart with miniature versions will not require any cutting of pieces, except possibly along the exterior edge of your tiling.
16. Prove that any valid tiling will still be valid after kites and darts are replaced with mini-kites and mini-darts.

## Tough Stuff

17. Prove that a tessellation cannot have more than one center of five-fold symmetry.

## More Stuff

These problems are from Fractions, Tilings, and Geometry by Kerins, Yong, Cuoco, Stevens, and Pilgrim, published in 2017 by the American Mathematical Society, available through your favorite online bookstores. The book was based on the PCMI morning mathematics course from 2014.

Why should pentagons get all the attention? Everybody loves the number 10 but nobody loves decagons.

What does this say about infinite tilings of the plane?



