Scaling the Teaching Curve:
A PCMI Weekend Experience
5 & 6 February 2016
University of Utah, Salt Lake City UT

Math block 1: Coins I

Take 1 or 2 (with 9 coins)

- Setup: 9 coins
- Legal moves: in each turn, take 1 or 2 coins
- Winner: takes the last coin

After explaining the game, ask the teachers to keep the following questions in mind: Does the first or second player have an advantage? Stronger, can the first or second player always win? (You may choose to delay the second question.) Lest anyone start with a general analysis, encourage them to begin by playing several rounds of the game.

Answer: The second player can always win (with 9 coins). His goal is to always leave a multiple of 3 stones. That is, if the first player takes 1, he takes 2; if she takes 2, he takes 1.

A common intermediate step is "getting down to" 3 or 4. In such cases, you may suggest that students start with 6 stones instead of 9. A frequent error is to focus on even and odd numbers of coins. False conclusions can be challenged by playing the game against a teacher—play to win! Even if you go first, the second player may make an error that allows you to leave him with a multiple of three. Play no more than one game with any group in a "visit," lest they figure out the winning strategy from your play rather than their analysis.

In describing the winning strategy, teachers may talk about taking "the opposite" number of coins. If so, then ask them to define opposite. The most succinct description, when presenting your opponent with a multiple of 3 coins, is to counter taking x coins by taking $3 - x$, leaving the next smaller multiple of 3.

Some pairs and tables may analyze the initial game quickly. Assign the following variations in order, as necessary to keep everyone engaged.

Take 1 or 2 (with 13 coins)

- Setup: 13 coins
- Legal moves: in each turn, take 1 or 2 coins
- Winner: takes the last coin

Give the team four more pennies. Now first player can win by taking 1 coin and then following the multiple of 3 analysis above. Ideally, each group gets through this case and perhaps the general number of coins described next.

With 9 and 13 coins correctly analyzed, ask the teachers to consider an arbitrary number of coins n. If $n \equiv 0 \bmod 3$, then the second player can win as with 9 coins above. If $n \not\equiv 0 \bmod 3$, then the first plater can win by taking $n \bmod 3$ coins and responding to the second player's moves as above.

The following two variations need not be considered by all participants.

Misère **Take 1 or 2** (with 13 coins)

- Setup: 13 coins
- Legal moves: in each turn, take 1 or 2 coins
- Loser: takes the last coin

Switching who wins and loses creates the misère version of a game, which can be more difficult to analyze. This one, however, is equivalent to the standard version of the 12 coin game: After the second player wins the 12 coin game, his opponent must take the final "poison" coin.

Take 2 or 3

- Setup: n coins
- Legal moves: in each turn, take 2 or 3 coins
- Winner: ???

This variation is not often needed. Teachers playing this should think about how to modify the notion of winning, since 1 coin could be left. When neither player can take the last coin, one might introduce a "draw" which is better than losing and worse than winning. A more standard approach is to give the win to whoever makes the last legal move (whether that leaves 0 or 1 coins). With this final convention, the second player can win with $n \equiv 0, 1 \bmod 5$ (respond to x with $5 - x$), while the first player can win when $n \equiv 2, 3, 4 \bmod 5$ by removing 2 or 3 coins to get to 0 or 1 mod 5 and playing as before.

Near the end of the block, call the room together and explain the following game from the early reality show *Survivor Thailand*.

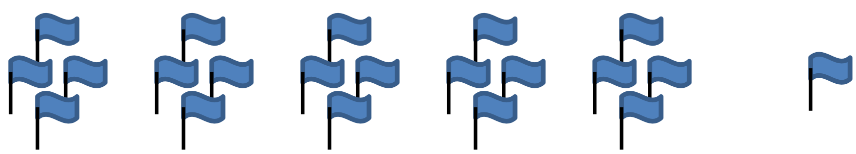
> **Take 1, 2, or 3** (with 21 coins)
>
> - Setup: 21 coins
> - Legal moves: in each turn, take 1, 2, or 3 coins
> - Winner: takes the last coin

Ask for two volunteers to play, with the other participants encouraging them and offering advice. If an overhead projector or document camera is available, then place 21 coins (in no order), have players stand on opposite sides, and move 1, 2, or 3 coins to their side at each turn so everyone can see play progressing. Otherwise, say what happens at each move, e.g., "Rita takes 2 coins leaving 14 total." The game may be repeated a few times as necessary.

Player 1 can always win by taking 1 coin. After her opponent takes x coins in his move, she responds by taking $4 - x$ coins, always leaving him a multiple of 4.

The 2002 television clip is available at `http://www.criticalcommons.org/Members/JJWooten/clips/survivor-21-flags`. No one seems to demonstrate any strategy until 6 flags remain.

Now introduce a visual representation to suggest the strategy. With an overhead projector or document camera, arrange the coins in 5 groups of 4 with 1 left over. Otherwise, draw that arrangement on a chalkboard, whiteboard, or pad of large paper on an easel.



Ask how this arrangement could assist players. Introduce the terminology and notation $21 \equiv 1 \bmod 4$. Ask for the analogous visual arrangements for the "take 1 or 2" games and how to express the solutions for that game in modular arithmetic.

These games are all restricted versions of an important combinatorial game called Nim which was named and fully analyzed in 1901. It allows for multiple piles of coins and the winning strategy depends on binary representations of numbers.

Math Block 2: Pythagorean Triples

Following up from the first block, start with addition and multiplication tables modulo 3 as a tool.

Pythagorean triples are integer solutions for $a^2 + b^2 = c^2$. Ask teachers for examples. As you record them, separate the primitive ones (where there is no factor common to all three values). In this block, participants will examine arithmetic properties of primitive Pythagorean triples. A complete list of such up with $c \leq 100$ is given on the next page, which can be copied as a handout.

Ask participants to look for patterns in the data. The following hold for the 16 triples provided; our work will be to show that they hold for all primitive Pythagorean triples.

1. One of the legs a, b is even. The other leg and c are odd.
2. One leg is a multiple of 3.
3. One leg is a multiple of 4.
4. One side length is a multiple of 5.
5. The hypotenuse is 1 more than a multiple of 4.

Verifying all of these requires modular arithmetic. Details are included in the following document, "Pythagoras on the Jersey Shore," which also documents another PCMI-related professional development program. The article appears in a monograph celebrating the centennial of AMTNJ, New Jersey's NCTM affiliate.

In later discussions, it is worth mentioning that rules such as "odd plus odd is even" is more succinctly stated as modulo 2 arithmetic. Also, some moduli have "nontrivial zero divisors," e.g., $2 \cdot 2 \equiv 0 \bmod 4$, which would complicate the methods of high school algebra. Here, the equation gives a situation where $a^2 \equiv 0 \bmod 4$ but $a \not\equiv 0 \bmod 4$.

Participants sometimes find other patterns, often for cases where there is a constant difference between the longest leg and hypotenuse, especially $c = b + 1$ and $c = b + 2$. The references in the article includes sources where additional patterns are explored.

Sometimes there are incorrect conjectures, which are still worthwhile pedagogically. For many of the given triples, one side is a prime, but this fails for $(33, 56, 65)$ where each side is composite. One conjectures here was if a leg is a multiple of 5, then the hypotenuse is prime: This is true for all 7 occurrences in the 16 given triples, but fails for $(119, 120, 169)$. Points worth making: A single counterexample suffices to disprove a conjecture, and the smallest counterexample may involve larger numbers.

Primitive Pythagorean triples
with hypotenuse less than 100

$$(3, 4, 5)$$

$$(5, 12, 13)$$

$$(8, 15, 17)$$

$$(7, 24, 25)$$

$$(20, 21, 29)$$

$$(12, 35, 37)$$

$$(9, 40, 41)$$

$$(28, 45, 53)$$

$$(11, 60, 61)$$

$$(33, 56, 65)$$

$$(16, 63, 65)$$

$$(48, 55, 73)$$

$$(13, 84, 85)$$

$$(36, 77, 85)$$

$$(39, 80, 89)$$

$$(65, 72, 97)$$

Pythagoras on the Jersey Shore

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A residential teacher professional development program with strong ties to the Association of Mathematics Teachers of New Jersey was held in Ocean Grove, New Jersey, from 2007 to 2011. We begin by describing the background and format of this Institute for New Jersey Mathematics Teachers, including its ties to the Institute for Advanced Study. For most of the article, we explore Pythagorean triples with modular arithmetic, part of the 2010 content.

Keywords: teacher professional development, Pythagorean triples, modular arithmetic

INTRODUCTION

For five years, the Institute for New Jersey Mathematics Teachers (INJMT) was held in Ocean Grove, New Jersey. Originally called the Ocean Grove Mathematics Institute, INJMT was a residential teacher professional development program inspired by the Institute for Advanced Study / Park City Mathematics Institute (IAS / PCMI) and its practices of group learning and content-based professional development. This article serves to document the program and give an example of the mathematic content covered.

INJMT traces back to the Summer School Teachers' Program of PCMI, a major annual institute for some 250 mathematics professionals focused on a different research theme each year. To support teachers throughout the school year, PCMI has established the New Jersey Professional Development and Outreach group (NJPDO). The NJPDO expanded to include teachers who had not participated in PCMI, and is still active. INJMT arose from the desire to provide New Jersey teachers an experience similar to PCMI, locally and on a smaller scale. The program enjoyed strong connections to the Association of Mathematics Teachers of New Jersey (AMTNJ). More details are given in the first section of this paper.

The trademark of PCMI and INJMT teacher professional development is a focus on mathematical content and group discovery learning in groups with minimal direct instruction. Although it is difficult to replicate that style in static prose, the second section gives an example from the 2010 program, *Explorations in Geometry!*

INSTITUTE FOR NEW JERSEY MATHEMATICS TEACHERS

A proper history of INJMT requires explanation of a few connected professional development programs. Although INJMT is presently inactive for want of funding, the programs described here are still active; teachers interested in them are encouraged to participate. Recording the format of INJMT offers tips for designers of professional development programs. We also highlight the connections between AMTNJ and INJMT.

Institute for Advanced Study and the Park City Mathematics Institute

Founded in 1930, the Institute for Advanced Study is "one of the world's leading centers for theoretical research and intellectual inquiry." Those who have worked at the Princeton facility (independent of Princeton University) include Albert Einstein, John Nash, and Freeman Dyson, among 33 Nobel Laureates and 38 Fields Medalists (a comparable honor for mathematicians).

In 1991, the University of Utah sponsored a Regional Geometry Institute in Park City, Utah, bringing together university researchers, graduate students, undergraduates, and teachers for three weeks of interconnected programs. IAS assumed sponsorship of the institute starting in 1993.

The Summer School Teachers Program involves some 60 teachers each year (always with representation from the NJPDO). Currently there is an emphasis on the Common Core State Standards in Mathematics and a partnership with Math for America (Clemens, 2012).

Among the many teacher activities at PCMI, we detail here the "morning math" course, facilitated by the Education Development Center. For two hours each day, teachers work in groups of six on a series of open-ended questions which weave together several mathematical topics. The approach encourages exploration, collaboration, and the work of making connections and constructing knowledge. Classroom-wide discussions occur only once or twice per hour, typically highlighting a participant's interesting approach for the larger group or summarizing a theme. There is almost no direct lecture instruction. While this pedagogical approach is sometimes unfamiliar or even uncomfortable for teachers, by the end of each institute the morning math class is uniformly rated as the teachers' favorite activity. These PCMI problem sets dating back to 2001 are freely available (http://mathforum.org/pcmi/hstp/problemsets.html). A formal publication, including the philosophy of this approach and tips for implementation, is in process. A related text was published by Al Cuoco of EDC in 2005.

New Jersey Professional Development and Outreach group

Professional Development and Outreach groups were founded to provide academic year support and activities for teachers who had attended PCMI. Many have expanded to include area teachers who have not attended the Utah program.

The NJPDO traces back to a group started in 2001 by David Keys at Rutgers University Newark. In early 2003, Saint Peter's University (College at the time) became the PDO sponsor and the author started working as the facilitator. Another PDO associated with Rider University also fed into the current group.

The NJPDO is centered on mathematical content explored via interactive group learning, with many sessions led by teachers. We meet at least four times per academic year, with midday sessions held on Saturday or Sunday. Most meetings are held at Saint Peter's University in Jersey City, and for the past few years Peddie School in Hightstown has hosted a spring meeting. Hands-on sessions on various mathematical content are led by teachers, the author, or by guests, who have included George Hart, Joseph O'Rourke, and Philip Mallison. We also enjoy "field trips" whose destinations have included the Liberty Science Center, the new National Museum of Mathematics, New York University's Institute for the Study of the Ancient World for an exhibit of Babylonian mathematical tablets, the Brooklyn Academy of Music for the Philip Glass opera Kepler, and the Museum of Chinese in America for an exhibit of puzzles, where NJPDO teachers helped the museum staff develop educational materials for schoolchildren visiting the exhibit.

Over one hundred teachers have participated in NJPDO activities, from across

New Jersey, New York City, Philadelphia, and even Ontario. Although the Ocean Grove program described in this article has ceased for the time being, the NJPDO still offers active programming to a vibrant community of teacher-leaders. We are always glad for more teachers to join us.

Details of Ocean Grove meetings

INJMT was held in the Lillagaard Bed and Breakfast near the boardwalk in Ocean Grove. The residential aspect of the institute was important, to allow participants to focus on the experience and spend time with each other; it was more like a retreat than work. Room and board were covered, and participants also received a stipend. Institutes were held mid-August; specific dates and programs are listed below.

12 - 17 August 2007, *Numbers! From Common Divisors to Cryptography*
10 - 15 August 2008, *Take the Number Train! Visualizing Pascal, Fibonacci, and More*
16 - 21 August 2009, *It All Adds Up!*
15 - 20 August 2010, *Explorations in Geometry!*
15 - 19 August 2011, *Math & Games*

A typical institute day started with breakfast downstairs, two morning sessions usually held on the second floor porch, lunch that was ordered in, two afternoon sessions, free time for enjoying Ocean Grove and finding dinner in town, and a hands-on evening session. The mathematical sessions were centered on open-ended problems explored in three groups of four teachers each. The facilitator (the author) would sometimes speak to the entire group to introduce or summarize a topic, but primarily worked individually with groups as questions arose, offering few answers and encouraging continued exploration. Evening activities included ZomeTools, Cuisenaire rods, the card game Set, slide-together geometric constructions, origami, MasterMind tournaments, combinatorial games, and a "math walk" through the neighborhood.

While the primary focus of INJMT was mathematical content for participants' own enrichment and life-long learning, several teachers reported modifying material they had learned for use in their classrooms. The format of working collaboratively with colleagues on open-ended problems with minimal direct instruction also had a strong impact. Teachers reported better understanding of their own students working on new material, a sense of empowerment as they constructed new knowledge, and better skills at facilitating group work in their own classrooms.

Connections with AMTNJ

While most funding for INJMT came from IAS / PCMI, AMTNJ supported the Ocean Grove program financially from 2008 to 2011. Each year, several of the twelve participants were AMTNJ members, often officers. In addition to mathematical enrichment, AMTNJ benefited by recruiting several new members and even officers from other INJMT participants. The current roster of active NJPDO teachers enjoys a broad overlap with AMTNJ membership, and hopefully the organizations will continue to collaborate and thrive together.

PYTHAGOREAN TRIPLES

A Pythagorean triple consists of three positive integers (a,b,c) satisfying $a^2 + b^2 = c^2$. The name comes from the geometric interpretation of these values being the side lengths of right triangles, so that their defining equation matches the consequent of the Pythagorean theorem. Following that connection, we will refer to a and b as the legs and c as the hypotenuse.

We will restrict attention to primitive Pythagorean triples, those where a, b, c share no common factor greater than one. For example, the triple (6, 8, 10) is not primitive, since each integer a multiple of 2 (even). The primitive Pythagorean triples with hypotenuse less than 100 are given in Table 1.

Table 1: The sixteen primitive Pythagorean triples with hypotenuse less than 100.

(3, 4, 5)	(5, 12, 13)	(8, 15, 17)	(7, 24, 25)
(20, 21, 29)	(12, 35, 37)	(9, 40, 41)	(28, 45, 53)
(11, 60, 61)	(33, 56, 65)	(16, 63, 65)	(48, 55, 73)
(13, 84, 85)	(51, 68, 85)	(39, 80, 89)	(65, 72, 97)

At the 2010 INJMT, teachers began this topic by making conjectures based on this list, which led to investigations of general results. The following discussion mirrors the explorations of those twelve teachers.

Parity

One of the first observations from Table 1 concerns the parity of a, b, c: In each case, one leg is odd, one even, and the hypotenuse is odd. Is this always the case, or could there be a larger primitive Pythagorean triple not following this pattern? Any finite amount of data is insufficient to resolve this question, calling for general techniques. Conjectures often follow from evidence, while proof requires careful logical argument applicable to all possible cases.

By the definition of primitive Pythagorean triples, it is not possible for all three numbers to be even. Two other cases are ruled out by basic facts about even and odd numbers, summarized here.

> **Parity Results**: An even number squared is even; an odd number squared is odd. The sum of two even numbers is even, the sum of two odd numbers is even, and the sum of an even and an odd number is odd. (Compare with Table 2 below).

From these facts, it is not possible for all three numbers to be odd, since then a^2, b^2 and c^2 would all be odd but $a^2 + b^2$ is even. Similarly, it is impossible for two of the numbers to be even and one odd.

We are left with the case that two of the numbers are odd and one is even. But the evidence suggests something more specific; in every case we have seen, the hypotenuse is odd. Is it possible to have both legs odd and the hypotenuse even? The parity results do not rule this out.

In order to address this case, we recast the parity results in terms of modular arithmetic, which will generalize in helpful ways. Arithmetic modulo k looks just at the remainders when dividing by k, and is thus restricted to the values 0, 1, …, k-1. A common example is "clock arithmetic," such as 5 hours after 10 o'clock is not 15 o'clock (at least in the civilian world), rather 3 o'clock, which can be determined by adding 10 and 5 modulo 12.

An even number is a multiple of two, and an odd number can be described as an integer with remainder one when divided by two. All of the parity results mentioned above are contained in the following addition and multiplication tables for arithmetic modulo 2, Table 2.

Table 2: Addition and multiplication modulo 2.

2 +	0	1
0	0	1
1	1	0

2 x	0	1
0	0	0
1	0	1

In order to consider whether both legs in a Pythagorean triple can be odd, one can consider the $a^2 + b^2 = c^2$ equation reduced modulo 4, whose arithmetic is shown in Table 3.

Table 3: Addition and multiplication modulo 4.

4 +	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

4 x	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

If *a* and *b* are odd, then we see in Table 3 that both a^2 and b^2 are 1 modulo 4. The sum $a^2 + b^2$ is then 2 modulo 4. Looking on the diagonal of the modulo 4 multiplication table, nothing squared is 2. Therefore $a^2 + b^2 = c^2$ with *a* and *b* odd has no solution modulo 4. A solution to $a^2 + b^2 = c^2$ in normal integers would reduce to a true statement modulo 4, so this reasoning shows that there is no Pythagorean triple with both legs odd.

Having ruled out every other possibility, we conclude that every primitive Pythagorean triple has one odd leg, one even leg, and an odd hypotenuse.

Multiples of 3, 5, and 4

There are more patterns evident from the data in Table 1. In each case, the even leg is moreover a multiple of 4. Also, one leg is a multiple of 3, and some number is a multiple of 5. How can we argue that these events occur for any primitive Pythagorean triple, no matter how large?

As the reader may guess by now, we will consider other moduli. Table 4 shows the entire arithmetic of integers modulo 3.

Table 4: Addition and multiplication modulo 3.

3 +	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

3 x	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Looking down the diagonal of the multiplication table, we see that any number squared is either 0 or 1 modulo 3. Notice also that only 0^2 gives 0. In the normal setting of all integers, this means that if an integer n has n^2 as a multiple of 3, then n must itself be a multiple of 3. Reducing the relation $a^2 + b^2 = c^2$ modulo 3 therefore has only a few possibilities. It could reduce to 0 + 0 = 0, but in that case, *a, b, c* are all multiples of 3, violating the definition of primitive. It is impossible for both a^2 and b^2 to be 1, since their sum 2 is not a square modulo 3. This leaves only 0 + 1 = 1 or 1 + 0 = 1 as possible reductions. In both cases, the 0 on the left hand side of the equations indicates that one of the legs is a multiple of 3.

The argument that one of the numbers in a Pythagorean triple must be a multiple of 5 follows similarly from modulo 5 arithmetic; see Table 5.

Table 5: Squares modulo 5.

n	0	1	2	3	4
n^2	0	1	4	4	1

The squares modulo 5 are 0, 1, and 4. Again, only 0^2 gives 0, so if n^2 is a multiple of 5, then n must be a multiple of 5. Possible reductions of $a^2 + b^2 = c^2$ modulo 5 are 0 + 0 = 0 (ruled out by primitivity), 0 + 1 = 1, 1 + 0 = 1, 1 + 4 = 0, and 4 + 1 = 0. Each of these includes a single 0, which means that one of the numbers must be a multiple of 5.

The same approach does not quite work for 4. For instance, even though $1^2 + 2^2 \neq 3^2$ in normal integers, we see from Table 2 that $1^2 + 2^2 = 1 + 0 = 1 = 3^2$ mod 4. The crux of the problem is that $2^2 = 0$ mod 4. This is where modular arithmetic can be strange: nonzero numbers can multiply together to give 0 in a composite modulus. This makes algebra in some modular arithmetic very different, for instance: just because the product of two expressions is 0 modulo a composite, one may not conclude that one or the other of them must be 0.

(Lest the reader think that this mathematical structure is too bizarre to be meaningful, realize that modular arithmetic is a foundational tool for computer science and information theory. The encryption used in every secure internet transaction is built on prime numbers and modular arithmetic. A good introduction to these applications is Kirtland (2001).)

The key insight for showing that each primitive Pythagorean triple contains a leg that is a multiple of 4 is to look at arithmetic modulo 8. (This may seem like a leap, but some of the 2010 NJPDO teachers came up with this idea.)

Table 6: Squares modulo 8.

n	0	1	2	3	4	5	6	7
n^2	0	1	4	1	0	1	4	1

As shown in Table 6, each odd number squares is 1 modulo 8, and an even number squared is either 0 or 4 modulo 8. Since one leg of a primitive Pythagorean triple must be odd and the other even, the only possible reduction of $a^2 + b^2 = c^2$ modulo 8 is either 0 + 1 = 1 or 1 + 0 = 1 (it is impossible to have the left hand side be 1 + 4 or 4 + 1 since 5 is not a square modulo 8). Since each possibility includes a single 0, from Table 6 we know the number being squared is either 0 or 4 modulo 8. That is, it is a multiple of 4.

Discussion and Further Reading

We summarize our work in the following theorem.

Theorem: If *a*, *b*, *c* are positive integers without a common factor greater than 1 that satisfy $a^2 + b^2 = c^2$, then:
i) one of *a*, *b* is odd, and the other is a multiple of 4;
ii) one of *a*, *b* is a multiple of 3;
iii) *c* is odd; and
iv) one of *a*, *b*, *c* is a multiple of 5.

Are there other divisibility results for primitive Pythagorean triples? The well-known triple (3, 4, 5) shows that there can be no larger divisor guaranteed for every triple. Also, examples from Table 1 show that any combination of the three guaranteed divisors is possible: (5, 12, 13) has a leg divisible by 3 and 4, then (8, 15, 17) has a leg divisible by 3 and 5, while (20, 21, 29) has a leg divisible by 4 and 5, and finally (11, 60, 61) has a leg divisible by all three necessary divisors.

There are many more patterns extant in primitive Pythagorean triples. A very accessible source is a book by Sierpinski (1972), better known for a fractal. More recent treatments connect Pythagorean triples to matrices (Hall, 1970) and complex numbers (Kerins et al., 2003, which also describes more about the PCMI teachers program).

While INJMT is presently defunct, its impact continues through the classroom practice of the program alumni and the ongoing teacher professional development work of IAS / PCMI, NJPDO, and of course AMTNJ.

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Math Block 3: Coins II

Review with the participants of the Take 1 or 2 game with 13 coins, then have them analyze the following game which has only a slight variation in the rules.

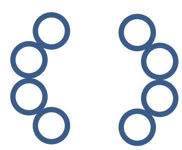
Daisy

- Setup: 13 coins in a tight circle (so that adjacent coins touch)
- Legal moves: in each turn, take any one coin or any two touching coins
- Winner: takes the last coin

Surprisingly, the adjacency requirement for taking pairs makes it a very different game.

For any number of stones, the second player can always win using a symmetric technique, an example of "strategy stealing." The first player takes 1 or 2 adjacent stones from the circle. The second player should remove 1 or two 2 at the opposite side of the circle in order to leave two isolated arcs having the same number of stones. (For the 13 coin game, in response to 1 or 2 being removed, player 2 should take the opposite 2 or 1, respectively, leaving two arcs of 5 stones each.) Then, whatever player 1 does in one arc, player 2 repeats in the other arc, as if there were a mirror on the board. This guarantees that player 2 will make the last legal move.

It is rare that teachers find this winning strategy. When you play, do play to win (see the notes below should you go first), but try not to be too obvious when mirroring. One intermediate step might be to ask a pair what happens should play come down to two arcs of 4 each.



Another hint could be to remind participants that one team in the 21 Flags video mentioned "doing what they did" as a strategy; that did not work there, but what could it mean in this game?

Participants are usually surprised at this "trick" and how different the solution is from the Coins I games.

Notice that after the first move, the circular aspect of the game is irrelevant—the remaining stones might as well be in a row rather than around a circle. Starting with stones in a row is the next game.

In actual play it is worth knowing how to win should player 2 not use this mirror approach. For notation, we write 12 for the daisy after one petal is removed leaving 12 adjacent petals. After player 1 moves, the game is at either 12 or 11. The strategy described for player 2 above is to then move to $5 + 5$, the two arcs of 5. (There are additional winning moves for player 2 from 11:

8 + 1, 7 + 2, and 6 + 3, although how to proceed in each case is not as easily described.)

The following table gives the moves that player 1 should make if player 2 does not leave 8 + 1, 7 + 2, 6 + 3, or 5 + 5 (if presented with one of these, player 1 should remove something from the larger pile, probably, and hope that player 2 doesn't know a winning strategy). Play is presented down to a symmetric state, where strategy stealing may be employed, or 1 + 2 + 3, which is another winning state (verify this: in two turns you can leave your opponent with one of the symmetric states 2 + 2, 1 + 1 + 1 + 1, or 1 + 1).

Player 2 from 12 or 11	Player 1	next Player 1
11, 6 + 5	5 + 5	
10, 9, 6 + 4, 5 + 4	4 + 4	
10 + 1, 9 + 2, 9 + 1	6 + 2 + 1	3 + 2 + 1 or 2 + 2 + 1 + 1
8 + 3, 8 + 2, 7 + 4, 7 + 3	4 + 3 + 2	3 + 3, 3 + 2 + 1, or 2 + 2 + 1 + 1

Kayles

- Setup: 1 coin, a space, and 11 coins in a tight row (so that adjacent stones touch)
- Legal moves: In each turn, take any one coin or any two touching coins
- Winner: takes the last coin

The name Kayles comes from a 14c. English slaughter of the French *quilles*, the term for bowling pins. The bowling idea is that you can knock over one pin or two pins that are next to each other. The game was developed in 1907 by Henry Dudeney.

While it might seem more natural to start with 13 coins in a row, that version has a direct symmetry strategy allowing Player 1 to win: she should take the middle coin, leaving 6 + 6, and then copy whatever her opponent does. The Kayles game described is essentially the misère version of 11 coins in a row.

While understanding the mirror strategy helps analyze this game and participants will probably have considered many possible states in thinking about Daisy, this game is surprisingly complicated and there is no "nice" solution. Below is a brute force solution of this case; a complete and rather intricate analysis was done in 1956.

Player 1 can win, although it requires a very specific move: take either the 4th or 8th pin of the 11, leaving 7 + 3 + 1. Player has 11 possible responses, which will require 4 possible moves from Player 1. The following table outlines all possibilities down to "mirror" states where strategy stealing may be employed, or a state described above.

Player 2 from $7+3+1$	Player 1	next Player 1
$7+3$, $4+3+2+1$	$4+3+2$	
$7+2+1$, $6+3+1$	$6+2+1$	
$7+1+1+1$	$3+2+1+1+1$	$3+2+1$, $2+2+1+1$, six 1s, or four 1s
$7+1+1$, $3+3+3+1$, $5+3+1+1$, $5+3+1$, $4+3+1+1$, $3+3+2+1$	$3+3+1+1$	

In actual play, it is also helpful to know how to win as the second player should your opponent not leave you with $7+3+1$. (If you do face $7+3+1$, take one pin, perhaps from the largest pile, and hope for opponent error.)

Player 1 from $11+1$	Player 2	next Player 2
11, $5+5+1$	$5+5$	
$10+1$, $9+1$, $8+2+1$, $7+2+1$	$6+2+1$	
$9+1+1$, $6+4+1$	$4+4+1+1$	
$5+4+1$	$4+2+2+1$	$2+2+1+1$
$8+1+1$, $6+3+1$	$3+3+1+1$	

Although it is worth learning these moves in order to effectively beat participants in this game, the major point is the contrast to the earlier games. The games from Coins I have a simple strategy based on modular arithmetic, Daisy has a surprisingly simpler strategy based on symmetry, but there is no elegant solution to Kayles. Consider inviting teachers to compare and contrast winning strategies for the different games. Some good points might include how small differences can have large effects and the fact that not every question has a nice answer.

Math Block 4: The Josephus Problem

Start with an activity outside or in a clear indoor space where all participants can stand in a circle. The facilitator stands in the center.

Duck, Duck, Die

- Have teachers count off from 1 with the charge to remember their numbers.
- Starting with 1, the facilitator counts off "duck, duck, die" (so participant #3 is the first to die). Continuing around the circle, count only survivors (so skip over #3 in all subsequent passes around the circle).
- The winner is the last teacher standing.

In practice, "dead" participants should put their arms over their chest or sit—they will want to continue to watch the progression, but it needs to be clear who should be skipped on subsequent rounds. When there are few living participants, it can be helpful to have them raise their hands.

The history of the Josephus problem is in the participant handout by Ensley. The question is to look for patterns in who is the final survivor for n people in the circle. Also of interest is the second to last participant.

Decrease the number of people in the circle (at each stage, have the highest numbered teachers leave the circle). Try to avoid going down one by one to keep some surprise in the pattern. Do at least the total number and values around one of the "breaks" such as 31 and 30; 21 and 20; or 14 and 13. Let participants know that they will analyze the situation in groups, so someone should record the data.

n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$D(n)$	1	2	2	1	4	1	4	7	1	4
n	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
$D(n)$	7	10	13	2	5	8	11	14	17	20
n	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
$D(n)$	2	5	8	11	14	17	20	23	26	29
n	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
$D(n)$	1	4	7	10	13	16	19	22	25	28

Back Inside

After three or so rounds, teachers return to their tables. They should fill in other values of $D(n)$ (perhaps sharing some work across tables while also double-checking results), although it is not necessary to go beyond the total number of participants. Foreshadowing an extension, you may suggest that they keep track of the order of deaths for each n.

The primary goal is to determine the pattern in the "duck numbers" $D(n)$. The recursive pattern is

$$D(n) \equiv D(n-1) + 3 \bmod n$$

(using n rather than 0 when that arises), an unusual situation where the modulus changes at each step. Beyond the initial data, you can see runs of $1, 4, 7, \ldots$ or $2, 5, 8, \ldots$; the subtlety comes at the "breaks" and whether the next survivor is 1 or 2 (break numbers here include $4, 6, 9, 14, 21, 31, 47$). The justification for the recursive formula is that after player 3 dies, it is the $n-1$ player game starting at 4, so all labels are increased by 3.

In case participants ask, there is not really an exact formula for $D(n)$. The development of a direct formula for killing every second person is in the worksheet. The following formula is true, but it has a catch.

$$D(n) = 3n + 1 - \left\lfloor \kappa \left(\frac{3}{2}\right)^{\lceil \log_{3/2} \frac{2n+1}{\kappa} \rceil} \right\rfloor$$

where $\lfloor\cdot\rfloor$ is the floor function, $\lceil\cdot\rceil$ is the ceiling function, and $\kappa = 1.6222705\ldots$. The problem is that computing digits of κ uses a recursive definition much like the modular solution above. Until there is an independent computation of κ, this formula is essentially a repackaging of the former one.

Two survivors

In one version of the puzzle, Josephus has a confidant who also wants to survive the intricate suicide pact. In what position should the confidant stand so that he and Josephus are the last two soldiers standing? (This is where recording the order of deaths would save work.)

n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$D(n)$	1	2	2	1	4	1	4	7	1	4
$C(n)$	–	1	1	4	2	5	1	4	7	10
n	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
$D(n)$	7	10	13	2	5	8	11	14	17	20
$C(n)$	2	5	8	11	14	1	4	7	10	13
n	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
$D(n)$	2	5	8	11	14	17	20	23	26	29
$C(n)$	16	19	22	1	4	7	10	13	16	19
n	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
$D(n)$	1	4	7	10	13	16	19	22	25	28
$C(n)$	22	25	28	31	34	1	4	7	10	13

The recursive pattern is the same, $C(n) \equiv C(n-1) + 3 \bmod n$, but knowing $D(n)$ (repeated in this table) does not seem to help one determine $C(n)$ in general. The breaks occur at different places, the "runs" between them are of different lengths, whether a run starts with 1 or 2 seems to be unrelated, etc. It is true that $D(n) - C(n)$ is constant during runs for both sequences.

A Different Skip Number

As another variation, one can compute the last survivor when killing every eighth soldier. We will call this the "potato number" as in the title of the handout.

n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$P(n)$	1	1	3	3	1	3	4	4	3	1
n	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
$P(n)$	9	5	13	7	15	7	15	5	13	1
n	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
$P(n)$	9	17	2	10	18	26	7	15	22	29
n	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
$P(n)$	6	14	22	30	3	11	19	27	35	3

The recursive pattern is $P(n) \equiv P(n-1) + 8 \bmod n$. Notice that the breaks are much more frequent and runs can start on several values (in the table, $1, 2, 5, 7$ at least—do all possible starting values occur?)

Additional Extensions

In case they are needed, here are two extension questions from the handout.

- Suppose in the game with 6 people, Josephus is person 1 but before the game starts, the Roman leader says, "Hey Joey, *you* pick the skip number." What should he say so that he is the last person left?

  Using a skip number of 60 will work for sure (see the next answer), but the smallest number that will work is 3.

- Is it possible for Josephus to always come up with a response to the previous question no matter how many people are originally in the circle?

  For the game with n people, using skip number k that is the least common multiple of the numbers in $\{1, 2, 3, \ldots, n\}$ is guaranteed to work, but there are typically much smaller values.

The following worksheet, "Exploring recursion with the Josephus Problem (Or How to play 'One Potato, Two Potato' for keeps)" by Doug Ensley and James Hamblin, covers mostly the same material in a more scaffolded way. It appeared in the collection *Resources for Teaching Discrete Mathematics: Classroom Projects, History Modules, and Articles*, edited by Brian Hopkins, published by the Mathematical Association of America, 2009.

Exploring recursion with the Josephus Problem (Or How to play "One Potato, Two Potato" for keeps)

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Summary

The Josephus problem is addressed in many discrete mathematics textbooks as an exercise in recursive modeling, with some books (e.g., [1] and [3]) even using it within the first few pages as an introductory problem to intrigue students. Since most students are familiar with the use of simple rhymes (like Eeny-meeny-miney-moe) for decision-making on the playground, they are comfortable with the physical process involved in this problem. For students who may wish to pursue this topic independently, [4] and [5] provide nice surveys and bibliographies, and the website [2] provides web-based tools for exploring the problem directly. The activities presented here are intended to be completed by students in a single class period early in the semester. We find that an opening student-centered problem can get the class involved and set a good tone for the semester. Moreover, we find that many issues arising from this particular problem can be built upon throughout the course. The next section provides some suggestions for connections to other parts of the course.

Notes for the instructor

The Josephus problem can be explored through role playing or through carefully constructed pencil and paper activities, depending on the amount of time one wishes to devote to it. We list below some of the things we discuss just before the activity as well as some of the contexts in which we have students revisit the problem later on.

- A good preliminary discussion on recursion can be initiated with the following problems.

  a. Pose the question, "What is $1+2+3+\cdots+19+20$?" This provides a good opportunity to share the creative idea of regrouping in order to sum 10 copies of 21 for a total of 210.
  b. Followup with the question, "What is $1+2+3+\cdots+20+21$?" Some students will try the regrouping trick, but at least one should point out that you can simply add 21 to the previous answer.
  c. This idea of using a "similar but simpler" problem that has been solved previously is the very essence of recursive thinking.

- The activities presented here have been written to be completed with paper and pencil, but with the investment of more time one can have students act out the roles. This is a good ice-breaking activity early in the semester, but it does take more time. Through role playing, students will discover for themselves issues like "We need to remember who was first," and "We need a system for describing who is the last one left."

- There will be several opportunities later in a discrete mathematics course when one can reprise the Josephus game as a source for exercises and motivational examples. Computer science courses often use this problem as an exercise in recursive programming or in maintaining circular linked lists. Hence, with some cooperation from a friendly computer science instructor, this problem can prove useful in more than one context.

  a. (Mathematical induction) In the Josephus problem with skip number 2, prove that for all integers $n \geq 0$, if the game starts with 2^n players, then the person in position 1 will be the last person left. (This uses induction with the induction step involving the one pass all the way around the circle for the first time in which the even numbered people are eliminated.)

  b. (Follow up) In the Josephus problem with skip number 2, if $0 \leq k < 2^n$ and the game starts with $2^n + k$ players, then the person in position $2k + 1$ will be the last person left. This is a non-inductive argument consisting of removing the first k (even numbered) people and then applying #1 to the remaining circle of size 2^n.

  c. (Binary representation of numbers) Define the cyclic left shift of a binary numeral b as the number obtained from shifting the leading (i.e., leftmost) 1 bit to the rightmost end of the numeral. For example, the cyclic left shift of the binary numeral 1001101 is the numeral 0011011, which is the same as 11011. Show that if $0 \leq k < 2^n$, then the cyclic left shift of the binary representation of $2^n + k$ is the binary representation of $2k + 1$. Hence, the cyclic left shift of a number m gives the last person left in the m person Josephus game with skip number 2. This gives an "application flavor" to the study of binary numbers that may make them more intriguing.

  d. (Modular arithmetic) When introducing modular arithmetic, an analogy can be made to the Josephus problem in which the original circle of people are numbered 0 through $n - 1$. In particular, the patterns within the tables of "last person left" all have the relationship "add k" but with the provision that the addition "wraps around the circle" to refer to the actual people.

References

[1] Ensley, D. E. and J. W. Crawley. *Discrete Mathematics: Mathematical Reasoning and Proof with Puzzles, Patterns and Games*, New York: John Wiley, 2006.

[2] Ensley, D. E. and J. W. Crawley. Companion website for *Discrete Mathematics*, at http://webspace.ship.edu/∼deensley/DiscreteMath/

[3] Graham, R., D. Knuth and O. Patashnik. *Concrete Mathematics*, Reading, MA: Addison-Wesley, 1994.

[4] Herstein, I. N. and I. Kaplansky. *Matters Mathematical*, New York: Chelsea Publishing Company, 1974.

[5] Schumer, P. D. "The Josephus problem: Once more around," *Mathematics Magazine* 75 (2002) 12-17.

Worksheet on Exploring recursion with the Josephus Problem (Or How to play “One Potato, Two Potato” for keeps)

Introduction

Ancient mathematics problems that still hold their own are always fun to play with. A particularly good one, which happens to be named for a first century historian, has its origins in the Jewish - Roman war. The historian Flavius Josephus was apparently trapped by the Romans in a cave with 40 fellow Jewish rebels. As good soldiers they decided on suicide rather than capture, so they formed a circle and agreed that every third person would be killed until no one was left.

Josephus and a friend were more keen on being captured than their colleagues, so they quickly found the spots to stand to ensure they were the two remaining at the end of the grisly proceedings. Hence, the mathematically inept suffered an untimely demise while Josephus and his friend lived to tell the tale.

This morbid story doesn’t seem like much of a game or puzzle, but it has the same basic structure (with terminal consequences) as the age old way of choosing someone from a group: the “one potato, two potato” algorithm. We will spend some time in class today playing this type of game and analyzing our results.

Analyzing the Josephus Problem

In general, when we play the “Josephus game,” there will be a certain number of people standing in a circle, and a “skip number” that tells us how many people to count before removing someone from the circle. In the classical example described above, the number of people is 41 and the skip number is 3.

Let’s look at a simpler example. This time, there will be only six people in the circle, but we will keep the skip number at 3. We’ll continue to play until there is only one person remaining. Let’s say the people, named Ann, Beth, Chris, Dave, Emma, and Fred, are arranged as shown in Figure 1.

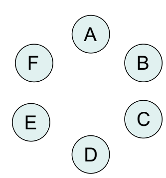

Figure 1: Six people in a circle

In this case, we decide to start counting with Ann. We count Ann and Beth, and when we get to the third person, Chris, he is removed from the circle. With Chris gone, we continue counting with Dave. We count Dave and Emma, and when we get to Fred, he is eliminated from the circle. Now there are four people left in the circle: Ann, Beth, Dave, and Emma, and the counting continues with Ann. We count Ann and Beth, and then Dave is eliminated. The current situation is displayed in Figure 2.

We next count Emma and Ann, and remove Beth, and the counting once again continues with Emma. We count Emma, Ann, and then Emma is removed, so Ann is the person who is left standing at the end.

An important thing to notice about this process is that we need to know which person to start

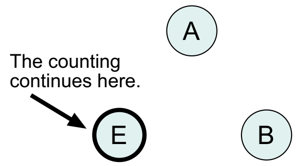

Figure 2: Three people remain

the counting with at each step, including the first step. If we remove a couple of people and then go on a coffee break, we might come back and forget who to resume the counting with.

> **For discussion**: Can you think of a way that we could remember which person we need to start the counting with at each step?
> *One solution is for us to put a funny hat on the person we need to start the counting with at each step. In our diagrams, we will put a thick circle around the "starting person."*

Let's try the game again, this time with seven people (named A, B, C, D, E, F, and G) and removing every fifth person. Recall that we say that the"skip number" is equal to 5. Figure 3 shows diagrams illustrating how such a game progresses. Note that the players are removed in the order E, C, B, D, G, A, and person F is the last one standing

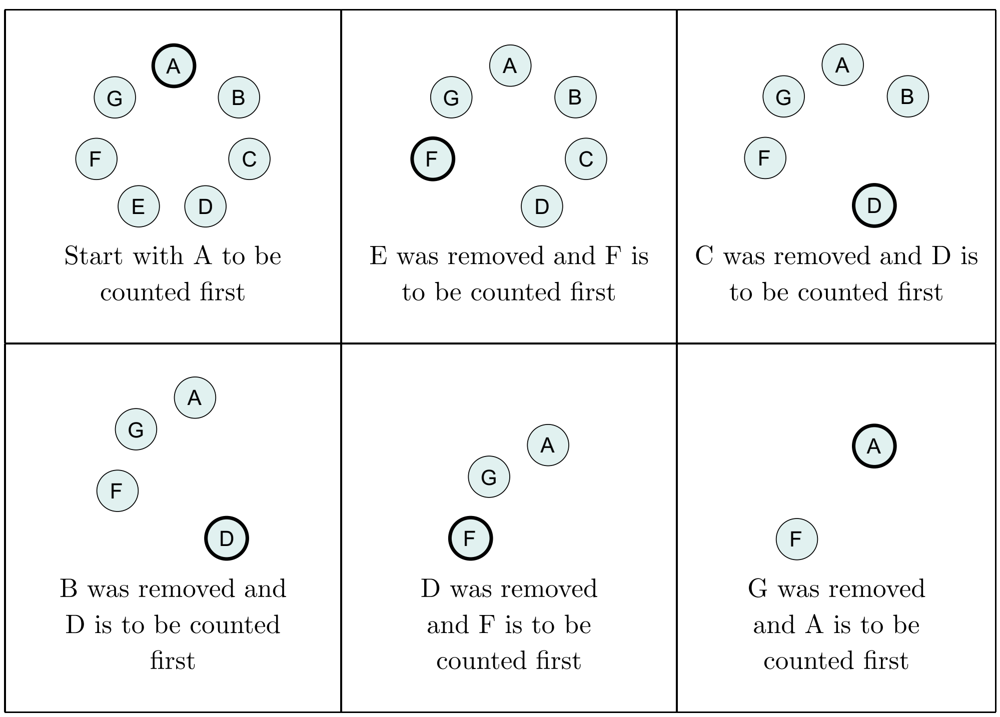

Figure 3: The game with seven people

Exercise 1 *On your own, play the Josephus game with n players and a skip number of k for each of the following values. Determine who is the last person standing.*
a. $n = 6, k = 2$ *b.* $n = 10, k = 3$ *c.* $n = 11, k = 3$

Changing the Starting Player

What happens if we decide to keep the values of n and k the same, but change the person we start the game with? How does this affect the outcome? Let's go back to the example with seven people and a skip number of 5. Let's say the people are named Terry, Ursula, Vivian, Walter, Xander, Yolanda, and Zack, and we want to start the game with Walter as Figure 4 shows.

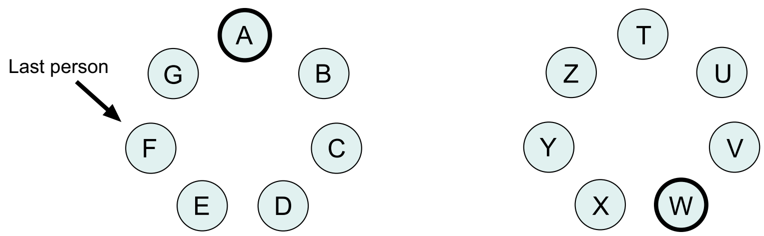

Figure 4: What's in a name?

> **For discussion**: In the game shown on the left in Figure 4, F is the last person standing. Who will be the last person standing in the game shown on the right? Can you figure it out without playing the entire game again?
> *If you said that Ursula would be the last person standing, you are correct! When we have seven people and a skip number of 5, the last person standing is the sixth one around the circle from the starting player. (Here we count the starting person as the first player around the circle.) In mathematical notation, we will write this as* $J(7, 5) = 6$.

The $J(n, k)$ notation is very handy for describing the last person left in the Josephus game that starts with n people in the circle and eliminates every k^{th} one. For example, the result of our first example can be described by simply writing $J(6, 3) = 1$.

EXERCISE 2 *Go back to the three games you played in Exercise 1. Using the mathematical notation we have defined, find the value of* $J(n, k)$ *in each of the following cases.*
a. $J(6, 2)$ *b.* $J(10, 3)$ *c.* $J(11, 3)$

Recursion: Using What Came Before

This idea of changing the starting player can be very helpful for finding patterns in the Josephus problem. Consider the game with eight people and a skip number of 5, as shown in Figure 5. After

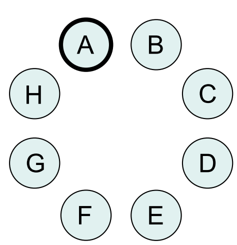

Figure 5: Beginning the game with eight people

the first step of this game, E is eliminated from the circle, and we have the situation in Figure 6.

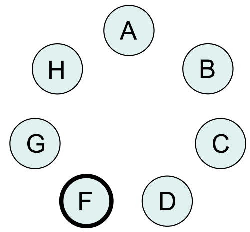

Figure 6: After person E is eliminated

Now what? Well, we continue to play the game as before, or we might notice that we have seen this situation before. This is a game with **seven** players and a skip number of 5. We already determined that the last person standing in this game is the sixth person around the circle from the starting player. In this case, that means that C is the last person standing.

For discussion: Finish playing the game to verify that C is the last one standing.

Here is another example of this idea. In Exercise 2(c), you determined that $J(11, 3) = 7$. That is, in a game with eleven people and a skip number of 3, the seventh person around the circle from the starting person will be the last one standing. How does this help us determine the value of $J(12, 3)$? Consider the first step of the game with twelve people and a skip number of 3. The first person eliminated is person 3, and person 4 becomes the new starting player. Now there are eleven people remaining, and we know that the last one standing will be the seventh person around the circle starting with person 4. This is person 10. You can verify on your own that $J(12, 3) = 10$.

Finding the Pattern

There is a pattern to how the position of the last survivor changes as we change the number of people initially standing in the circle. To see this pattern, we need to experiment and compute the answer for many different examples. In the table in Exercise 3, the top row shows the number of people in the circle, and the bottom row shows the position of the last person standing when the skip number is 3. The values we have already determined are filled in for you.

EXERCISE 3 *Fill in the rest of the table, either by playing each game or by appealing to the "using-what-came-before" strategy.*

n	***3***	***4***	***5***	***6***	***7***	***8***	***9***	***10***	***11***	***12***	***13***	***14***
$J(n, 3)$				*1*					*7*	*10*		

a. *What pattern do you notice in the table?*

b. *Can you explain in terms of the "using-what-came-before" strategy why this pattern holds?*

c. *On your own, make a similar table but change the skip number to 4. Can you predict what pattern you will see?*

An easier variation

A game that's a little better suited for detailed analysis is the variation where every second person is eliminated — that is, the skip number is 2. The game will officially be played with people named $1, 2, \ldots, n$ in a circle (with the numbers going clockwise). We go around the circle clockwise getting rid of every second person (Person 2 is the first to go) until no one is left. For example, if we start with four people, then the people are eliminated in the order $2, 4, 3, 1$, so person 1 is the last survivor.

We will let $J(n)$ denote the last survivor in the game which starts with n people and has a skip number of 2. (That is, we use $J(n)$ instead of $J(n, 2)$.)

Exercise 4 *Fill in the rest of the table, either by playing each game or by appealing to the "using-what-came-before" strategy.*

n	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*
$J(n)$				*1*												

n	*17*	*18*	*19*	*20*	*21*	*22*	*23*	*24*	*25*	*26*	*27*	*28*	*29*	*30*
$J(n)$														

a. *How is the value of $J(n)$ related to the value of $J(n-1)$?*

b. *What will be the next value of n for which $J(n) = 1$?*

c. *How would you describe a formula for $J(n)$ that would allow someone to quickly figure out the last place in line given any n?*

Josephus and his buddy

In the original story, Josephus actually escapes with a friend, so in reality he had to know the positions of the last two survivors of this macabre game. To keep it simple, let's still use the game with skip number 2, but now we will use $F(n)$ to denote the required position of the friend in the Josephus game starting with n people.

Exercise 5 *Play the Josephus game (with every second person eliminated, as above) for various n and record the numbers $J(n)$ and $F(n)$ of the last person alive and of the next-to-the-last person alive, respectively. Find more values than in the table below if you think it is helpful to do so. Remember to try to use things you already know as you tackle larger and larger values of n.*

n	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*	*21*	*22*	*23*	*24*
$J(n)$													
$F(n)$													

a. *How is the value of $F(n)$ related to the value of $F(n-1)$?*

b. *What will be the next value of n for which $F(n) = 1$?*

c. *Is there a direct relationship between $J(n)$ and $F(n)$?*

Further questions for exploration

The following problems, as well as the ones above, can be explored with the applet found under Section 1.1 on the website

http://webspace.ship.edu/∼deensley/DiscreteMath/flash/

EXERCISE 6 *Fill in the following table using the "One potato, two potato" game on n people, starting the first "one potato" on person 1. For those not familiar with this method of choosing a person on the playground, this is simply the Josephus problem with every* **eighth** *person eliminated. That is, in the table below we use $P(n)$ to mean the same thing as $J(n, 8)$ from the previous discussion.*

n	***10***	***11***	***12***	***13***	***14***	***15***	***16***	***17***	***18***	***19***	***20***	***21***	***22***	***23***	***24***
$P(n)$															

a. *If the ______ students in this class stand in a circle in alphabetical order and do "one potato, two potato", who will be the last person left?*

b. *Suppose in the game with 6 people, Josephus is person 1 but before the game starts, the Roman leader says, "Hey Joey,* you *pick the skip number." What should he say so that he is the last person left?*

c. *Is it possible for Josephus to always come up with a response to the previous question no matter how many people are originally in the circle?*

Solutions

Exercise 1. We will use the conventions of labeling the people A, B, C, etc. clockwise around the circle and starting our count with person A.

a. For $n = 6$ and $k = 2$, the last person left is E.

b. For $n = 10$ and $k = 3$, the last person left is D.

c. For $n = 11$ and $k = 3$, the last person left is G.

Exercise 2.

a. $J(6, 2) = 5$

b. $J(10, 3) = 4$

c. $J(11, 3) = 7$

Exercise 3. Here is the completed table:

n	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
$J(n, 3)$	2	1	4	1	4	7	1	4	7	10	13	2

a. For all $n \geq 2$, person $J(n, 3)$ is three more around (clockwise) the original circle from person $J(n-1, 3)$.

b. If the k^{th} person around the circle of $n-1$ people is the last one remaining, then in the game that starts with n people, after one person is eliminated the first person in the remaining circle of $n-1$ is person 4. The k^{th} person in *this* circle, is the $(k+3)^{th}$ person in the original circle.

c. For all $n \geq 2$, person $J(n, 4)$ is four more around (clockwise) the original circle from person $J(n-1, 4)$.

Exercise 4. Here is the completed table:

n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
$J(n)$	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1

n	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
$J(n)$	3	5	7	9	11	13	15	17	19	21	23	25	27	29

a. For all $n \geq 2$, person $J(n)$ is two more around (clockwise) the original circle from person $J(n-1)$.

b. The next value of n for which $J(n) = 1$ will be $n = 32$. It appears that $J(n) = 1$ if and only if n is a power of 2.

c. Given n people originally, let m be the smallest power of 2 less than or equal to n. Eliminate people $2, 4, \ldots, 2(n-m)$. This leaves the game with m people, the first of whom is person $2(n-m)+1$. According to the observation in part (b) of this exercise, this person will be the last person left at the end of the entire process.

EXERCISE 5. Here is the completed table:

n	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**
$J(n)$	9	11	13	15	1	3	5	7	9	11	13	15	17
$F(n)$	1	3	5	7	9	11	13	15	17	19	21	23	1

a. For all $n \geq 2$, person $F(n)$ is two more around (clockwise) the original circle from person $F(n-1)$.

b. The next value of n for which $F(n) = 1$ will be $n = 48$. It appears that $F(n) = 1$ if and only if $n = 3 \cdot 2^k$ for some value of $k \geq 0$.

c. $J(n) - F(n) = 2^k$ when the integer k can be chosen so that $3 \cdot 2^{k-1} \leq n < 2^{k+1}$, and $F(n) - J(n) = 2^k$ when the integer k can be chosen so that $2^{k+1} \leq n < 3 \cdot 2^k$.

EXERCISE 6. Here is the completed table:

n	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**
$P(n)$	1	9	5	13	7	15	7	15	5	13	1	9	17	2	10

a. This answer will depend on the number of people in your class. Suppose there are 32 people in your class. Using the pattern of "adding 8" relative to the number in the circle, we find that $J(32, 8) = 17$.

b. Using a skip number of 60 will work for sure (see the next answer), but the smallest number that will work is $k = 3$.

c. For the game with n people, using k that is the least common multiple of the numbers in $\{1, 2, 3, \ldots, n\}$ is guaranteed to work, but there are typically much smaller values.