

Rotation in $\mathbb{R}^2$: An Étude

Benjamin J. Hyman

bjhyman@cps.edu

Walter Payton College Prep, Chicago, Illinois

Park City Math Institute

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The goal of this étude is to link together a number of topics which receive short stint in the typical high school curriculum. You'll draw connections between the seemingly disparate subjects of complex numbers, matrices, and rotations of points and objects in the plane. Essential prior knowledge comprises only trigonometry on the coordinate plane, familiarity with geometric transformations, and elementary algebra skills. The first three sections can be completed in any order. Enough summary: let's dive right in.

1 Complex numbers ($\mathbb{C}$)

First, a note of assurance. Just because we attach words like “complex” and, later on, “imaginary” to this first subject doesn’t mean that the ideas herein don’t exist, or that you can’t get a handle on them. It’s all perfectly sound as an algebraic exploration.

Definition 1. Define i to be the quantity such that $i^2 = -1$.

You won’t be able to mark i on the number line you learned in elementary school, but that’s okay. We’ll refer to such “elementary school number line numbers” as “real numbers,” or elements of $\mathbb{R}$.

In other respects, i behaves in the same way as other (“real”) numbers you’ve encountered thus far. We’ll give some examples rather than belabor the point.

Example 1.

- $2i + 3i = 5i$
- $3i \cdot 4 = 12i$

- $1 + 2i = 1 + 2i$ (nothing else to be done!)
- $5i \cdot 4i = 5 \cdot 4 \cdot i \cdot i = 20 \cdot -1 = -20$

A good analogy for how to perform algebra with quantities involving i : treat such problems in the same way as you would those problems that involve other incommensurate values, e.g. $\sqrt{3}$ popping up in a problem where you're otherwise working with integers. That is, you could imagine replacing any instance of i in the examples above with $\sqrt{3}$ and the operations should still make sense.

Remark. For the sake of formalism, we'll note that the earlier sentence " i behaves in the same way as other ('real') numbers you've encountered" really means that multiplication and addition are still associative and commutative, the distributive property is upheld, plus a few other properties satisfied. And let's say no more about that.

Problem 1. We defined i as *the* quantity such that $i^2 = -1$, but there's actually another such value. Show that the quantity $-i$ also satisfies definition 1.

(So, like other numbers, i has an additive inverse $-i$ where $i + -i = 0$. It just so happens that this additive inverse also satisfies the same key property that i does.)

Problem 2. Evaluate the following.

- a) $5 \cdot (3 + i)$
- b) $(6 + i) - (3i - 4)$

Problem 3. Show that $3i \cdot 3i = -9$.

Remark. A common mistake is to assert that instead of -9 , one obtains for the product -3 .

Problem 4. Confirm in the first and second case, evaluate in the latter two.

- a) $(7 + 2i)(i) = -2 + 7i$
- b) $(1 + 2i)(3i - 4) = -10 - 5i$

c) $(7 + 2i)(3i - 5)$

d) $(\sqrt{3} - i)(\sqrt{12} + i)$

Definition 2. Given our above definition of i , we'll now consider *the set of complex numbers* $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$. A given complex number $z = a + bi$ has *real component* a and *imaginary component* b .

Any real number x can be conceptualized as a complex number $x + 0i$. Therefore, we can say $\mathbb{R} \subset \mathbb{C}$, i.e. $\mathbb{R}$ is contained in $\mathbb{C}$.

Problem 5. Evaluate $(a + bi)(x + yi)$. What is the real component of this quantity? The imaginary component?

1.1 Some optional extensions involving i

Problem 6. Evaluate each of the following.

a) i^3

b) i^4

c) i^7

d) i^{10}

e) Keep going. Or rather, generalize.

Problem 7. Factor each of the following.

a) $x^2 - 9$

b) $x^2 - 10$

c) $x^2 + 1$

d) $x^2 + 9$

Problem 8. Solve the following two quadratics.

a) $x^2 + 6x + 8 = 0$

b) $x^2 + 6x + 12 = 0$

Previously, you would have said that problem 8a has two solutions and 8b has no solutions. But really, you meant that 8b has no *real* solutions ...in fact, it has two (non-real) complex solutions, which you hopefully found above. Likewise with problem 7: every binomial there is factorable if we consider that we're factoring over $\mathbb{C}$ instead of over $\mathbb{R}$.

2 Matrices

The following is an example of a dimension 4 x 3 matrix, so-called because it has 4 rows and 3 columns.

$$\begin{pmatrix} 6 & 1 & 2 \\ 5 & 7 & 0 \\ 3.6 & 8 & \pi \\ \sqrt{2} & 4 & 7 \end{pmatrix}$$

At a first glance, matrices are nothing more than rectangular arrays of numbers. But they pack a surprising amount of depth, almost none of which we'll comment on here.

Let's consider a general 2×2 matrix. A matrix such as this, having the same number of columns as rows, is called *square*. We denote the element in row 1 and column 2 by a_{12} .

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

We can add two matrices of the same dimension together in a very natural ("component-wise") way, and obtain as a result a matrix of the same dimension:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}$$

We can also multiply a matrix by an element of $\mathbb{R}$ or $\mathbb{C}$, again in a natural way. Such a multiplier is called a “scalar.” In this case, we’ll consider a scalar of 7:

$$7 \cdot \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 7a_{11} & 7a_{12} \\ 7a_{21} & 7a_{22} \end{pmatrix}$$

2.1 Matrix Multiplication

Though valuable operations, we won’t really consider either addition or scalar multiplication. Rather, we’ll move on to our final operation: multiplying two matrices together.

Matrix multiplication occurs, unlike addition and scalar multiplication, in an apparently ad hoc way, not at all in the previously seen component-wise fashion. If you’ve never encountered it before, it’s off-putting, even bizarre. Trust that there are good and deep reasons for why we define it in the following manner.

Definition 3.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

That is, to find element c_{ij} in the product $A \cdot B$ (recall, c_{ij} is the element in row i and column j), you multiply the 1st element in row i of matrix A by the 1st element in column j of matrix B, and add to it the product of the 2nd element in row i (matrix A) and the 2nd element in column j (matrix B), and if we were multiplying matrices larger than 2×2 , we could continue, i.e. also add the 3rd element in row i times the 3rd element in column j , etc.

Here’s a more concrete example:

Example 2.

$$\begin{pmatrix} 7 & 5 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} = \begin{pmatrix} 7 \cdot 2 + 5 \cdot 6 & 7 \cdot 4 + 5 \cdot 8 \\ 3 \cdot 2 + 1 \cdot 6 & 3 \cdot 4 + 1 \cdot 8 \end{pmatrix} = \begin{pmatrix} 44 & 68 \\ 12 & 20 \end{pmatrix}$$

We find it helpful to place a left finger along row i of matrix A , and a right finger along column j of matrix B , and move our left finger rightwards and our right finger downwards simultaneously. Whenever both fingers are on two matrix entries (one from A and one from B), multiply those entries together, keeping a running sum of your total. That total is recorded in row i , column j of your product matrix.

Problem 9. Confirm that the following are correct.

$$\text{a) } \begin{pmatrix} 1 & 4 \\ 6 & 9 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 1 \cdot 2 + 4 \cdot 5 & 1 \cdot 0 + 4 \cdot 7 \\ 6 \cdot 2 + 9 \cdot 5 & 6 \cdot 0 + 9 \cdot 7 \end{pmatrix} = \begin{pmatrix} 22 & 28 \\ 57 & 63 \end{pmatrix}$$

$$\text{b) } \begin{pmatrix} 4 & 6 \\ 2 & 7 \end{pmatrix} \cdot \begin{pmatrix} 8 & 1 \\ -3 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 22 \\ -5 & 23 \end{pmatrix}$$

When multiplying two scalar numbers together, say 6 and 4, it's true that $6 \cdot 4 = 4 \cdot 6$.

Let's check to make sure the same is true for matrices.

Recall that we already have example 2, where $A = \begin{pmatrix} 7 & 5 \\ 3 & 1 \end{pmatrix}$, and $B = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$, and we calculated $A \cdot B$.

Problem 10. Calculate the 2×2 matrix $B \cdot A$.

If you did the above problem (and if you didn't, why are you reading this?!), you discovered the surprising fact that $A \cdot B \neq B \cdot A$. Matrix multiplication, unlike most of what you've probably seen in mathematics up to this point, is *non-commutative*.

Problem 11. Matrices don't need to be square (or of the same dimension) in order for you to multiply them together. Consider an $m \times n$ matrix A and a $p \times q$ matrix B . What relations must exist among $\{m, n, p, q\}$ in order for the product $A \cdot B$ to exist? What will the dimensions of that product be? Likewise, consider both questions in

the case of $B \cdot A$. Hint: We find the above-mentioned “finger-sliding” technique to be helpful in considering this problem.

Problem 12. A heart-pounding number of matrix multiplication exercises. Complete as many as strike your fancy. Some of these concepts will come up again.

i. $\begin{pmatrix} 0 & 3 \\ 4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 7 & 6 \\ 7 & 2 \end{pmatrix}$

ii. $\begin{pmatrix} \pi & 9 \\ \sqrt{2} & 11 \end{pmatrix} \cdot \begin{pmatrix} 3 & -2 \\ -5 & 1 \end{pmatrix}$

iii. $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

iv. Confirm that

$$\begin{pmatrix} 7 & 1 & 2 \\ -11 & 0 & 13 \\ 8 & 2 & 9 \end{pmatrix} \cdot \begin{pmatrix} 8 & 3 & 6 \\ 14 & 1 & -15 \\ 2 & 4 & -9 \end{pmatrix} = \begin{pmatrix} 74 & 30 & 9 \\ -62 & 19 & -183 \\ 110 & 62 & -63 \end{pmatrix}$$

v. $\begin{pmatrix} 2.5 & 9 & 1 \\ 0 & -4 & 3 \\ 10 & 6 & 8 \end{pmatrix} \cdot \begin{pmatrix} -8 & 0 & 11 \\ 4 & 5 & 7 \\ 2 & 1 & -12 \end{pmatrix}$

vi. $\begin{pmatrix} 1 & 4 \\ 6 & 9 \end{pmatrix} \cdot \begin{pmatrix} v \\ w \end{pmatrix}$

vii. Find matrix A such that $A \cdot \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 5v - w \\ 2v + w \end{pmatrix}$

viii. Find matrix A such that $A \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$

ix. Find matrix I such that $I \cdot \begin{pmatrix} 3 & 7 \\ 9 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ 9 & 1 \end{pmatrix}$. This I is known as the *identity matrix*.

- x. Find matrix B such that $\begin{pmatrix} 3 & 7 \\ 9 & 1 \end{pmatrix} \cdot B = \begin{pmatrix} 22 \\ 46 \end{pmatrix}$
- xi. Find matrix A such that $A \cdot \begin{pmatrix} 3 & 7 \\ 9 & 1 \end{pmatrix} = \begin{pmatrix} 12 & 8 \\ 6 & 16 \end{pmatrix}$

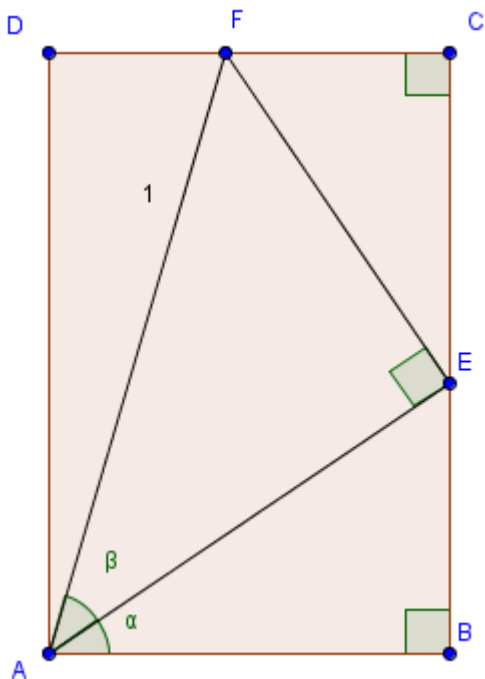
3 Angle Addition Formulae

Our goal is to prove the following angle addition formulae, which we will end up using later:

Theorem 1. $\sin(\alpha + \beta) = \sin(\alpha) \cos(\beta) + \sin(\beta) \cos(\alpha)$

Theorem 2. $\cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)$

Consider the following picture. Imagine that we begin with right triangle $\triangle AEF$ with a hypotenuse of length 1 and an angle β . Construct right triangle $\triangle AEB$ with angle α , and then construct the rest of $ABCD$. We won't prove that $ABCD$ is a rectangle, though we will rely on this fact.



Problem 13. Determine all unlabeled angles in the figure.

Problem 14. Determine the lengths of AE and EF .

Problem 15. Given that AE and EF are hypotenuses of their own right triangles, determine lengths AB, BE, EC , and CF .

Problem 16. Determine lengths AD and DF .

Problem 17. What do you know about the lengths of the opposite sides of a rectangle? Synthesizing this with your work on the previous problems, deduce the angle addition formulae. Awesome.

4 Transformations

Consider the common transformations: translations, dilations (scaling), reflections, and rotations. We could explore these geometrically, i.e. by sketching a coordinate plane, a particular object (a *preimage*), and the transformed object (*image*). Then we'd analyze how the transformation changed the preimage. But let's take a different tack, a more algebraic one. Instead of looking at the plane, we'll consider functions "transforming" points in accordance with particular rules.

Problem 18. Function T takes point (x, y) and transforms it by $T(x, y) = (3x, 3y)$. Describe, using the language of transformations, what exactly T does.

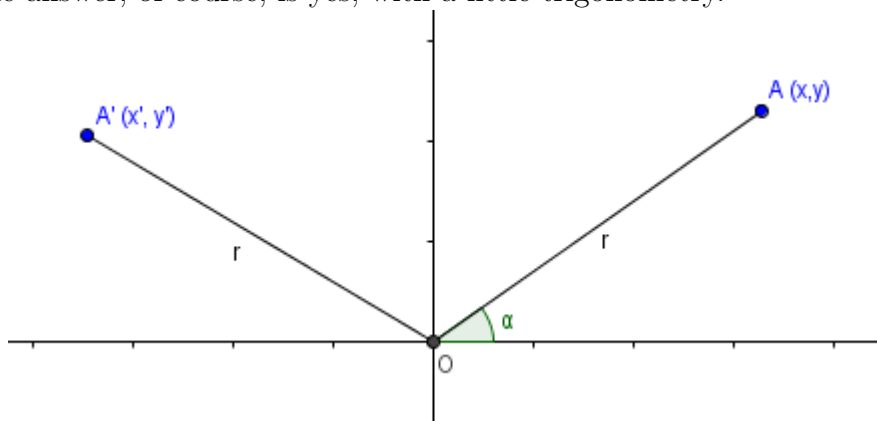
Remark. You may find it helpful to graph a series of points or a distinct shape, apply T to each point in that preimage, and inspect the results.

This is powerful stuff. Through T , we're systematically changing the coordinates of every point in our original space, and sending them elsewhere. (Except for the "fixed points" ... are there any here?)

4.1 Rotations

Is it possible to describe rotations like this as well? That is, given a fixed angle of rotation β , can we construct a transformation rule T that sends point $A(x, y)$ to its

image, point $A'(x', y')$, which is simply point A rotated about the origin β degrees? The answer, of course, is yes, with a little trigonometry.



Problem 19. Given that segment OA has length r , use trigonometry to express (x, y) in terms of r and α .

Problem 20. In the above figure, sketch in our angle of rotation β .

Problem 21. Use trigonometry to express (x', y') in terms of r , α , and β .

Problem 22. Your answer from problem 21 likely contains trig functions with arguments $(\alpha + \beta)$. Apply theorems 1 and 2, i.e. the angle addition formulae, to these expressions. Then, clean up this result, using your work from problem 19 (i.e. by substituting in those expressions with x or y as appropriate).

Problem 23. Summarize your findings in the form $T(x, y) = \dots$

Theorem 3. *Congratulations! You just discovered rotation by angle θ (well, β) as a transformation $T(x, y) = (\quad)$. Write it here for posterity.*

Problem 24. Using theorem 3, determine the transformation T that will rotate points by 90° . Check visually: graph points, apply T , and confirm by eyeballing/measuring that a 90° rotation indeed occurred. You could also use dynamic geometry software such as GeoGebra to verify this.

Problem 25. Convince yourself, without the use of theorem 3, that $W(x, y) = (-x, -y)$ accurately encapsulates a 180° rotation. (E.g. graph a triangle, reflect it over both axes,

and study the resulting picture for a few seconds.) Then, observe that plugging $\theta = 180^\circ$ into theorem 3 yields transformation W.

4.2 Rotations and matrices

Let's re-cast our observation from theorem 3 in matrix form. For a better sense of what that means, take a look at the next problem.

Problem 26. R is a matrix such that $R \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$. What must the dimensions of R be?

Here, we think of our points (x, y) and (x', y') as *column vectors* like seen above. Then we look for a matrix R such that R times our preimage (column vector) sends us to our image (column vector).

Problem 27. Find R by filling in the blanks: $\begin{pmatrix} - & - \\ - & - \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$

$\left[\text{Recall from theorem 3 that } \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \cos(\theta) - y \sin(\theta) \\ x \sin(\theta) + y \cos(\theta) \end{pmatrix} \right]$

Theorem 4. *Congratulations! You just constructed a (the) rotation matrix R . Write it here for posterity.*

Problem 28. Construct the matrix that rotates by angle θ and dilates by scale factor r . You may want to refer back to problem 18.

5 Complex numbers as vectors in $\mathbb{R}^2$

(If you haven't done so already, complete section 1. Back? Good.)

To tie what we've seen about complex numbers, matrices, and rotation together, we take the simple step of re-conceptualizing our complex number $a + bi$ as a coordinate pair (a, b) , and from there to column vector $\begin{pmatrix} a \\ b \end{pmatrix}$. The real component will be in row 1 of our vector, the imaginary in row 2. The coordinate pair, which was originally a complex number, can be drawn in $\mathbb{R}^2$, hence the title.

Problem 29. What quadrant is $3 - 2i$ in?

Problem 30. Where is $-i$ on the coordinate plane?

Definition 4. The coordinate plane on which we're drawing complex numbers isn't technically the Cartesian coordinate system you grew up with. It's known as the *complex plane* or *Argand plane*.

Problem 31. Which axis is the real axis, and which the imaginary?

Definition 5. Given a complex number $z = a + bi$, the quantity $a^2 + b^2$ is the *magnitude* of z .

Problem 32. Find the magnitude of $7 + 29i$.

Problem 33. Pick a complex number w and graph it in the complex plane. Construct the line segment between the origin and w , and determine its length. What do you notice?

6 Complex numbers and matrix multiplication

We'll first motivate this with a problem.

Problem 34. Graph $z = 2 + 3i$ in the complex plane. Then, express $z' = i \cdot z$ as a complex number in the form $a + bi$ and graph it in the complex plane. What's the relation between z and z' ?

Problem 35. As above, produce other complex numbers w and consider graphically the resulting $i \cdot w$.

We already explored how one rotates points (x, y) by an angle θ . It's extremely natural to perform the same rotation on complex numbers rather than points. (Of course, a complex number is more or less a point anyway.)

Problem 36. Determine the image of $1 + i$ under the following rotations.

- a) 30°
- b) 60°
- c) 90°
- d) 120°

Recall problem 5. Complete it now if you haven't done so already.

Problem 37. In the manner given in section 5, re-cast the following quantities from problem 5 as 2×1 column vectors.

- a) $(x + yi)$
- b) The evaluated product $(a + bi)(x + yi)$

Problem 38. Let P be the 2×1 column vector you found in 37a and S what you found in 37b. Much as you did in problem 27, find matrix M such that $M \cdot P = S$.

Remark. Just as our vector P above is entirely determined by the quantity $x + yi$, M is wholly determined by the multiplier $a + bi$. If we say $z = a + bi$, it's appropriate to denote M as M_z to indicate the relationship between M and z .

Theorem 5. *In a shameless attempt to inflate the number of theorems in this work, let's call your result from problem 38 a theorem. That is, multiplying a given complex number by $z = a + bi$ is equivalent to multiplying by matrix M_z , where M_z (you know the drill ... for posterity ...) =*

Where does this leave us? We've seen two transformation matrices thus far: the one from theorem 4 for rotating points by an angle θ , and the one just now, from theorem 5, for multiplying complex numbers by a fixed $a + bi$. Our final goal is to discover how the two relate to each other.

Problem 39. Choose a specific (a, b) such that $a^2 + b^2 = 1$, i.e. (a, b) is on the unit circle.

- a) Graph $z = a + bi$ in the complex plane. Observe that z has magnitude 1.
- b) Determine the angle ω that z makes with the positive x-axis.
- c) Choose a specific $V(x, y)$ in the plane $\mathbb{R}^2$. Denote the complex number $x + yi$ by v .
 - i) Using your matrix from problem 38, multiply v by z from above.
 - ii) Using your matrix from theorem 4, rotate point V by ω .
 - iii) What do you notice?

Let's formalize that noticing you just did.

Theorem 6. *Given a complex number z with magnitude 1, multiplying $x + iy$ by z is "the same" as rotating (x, y) by _____.*

This theorem is another way of remarking on the fact that, when our complex number is on the unit circle, the two types of rotation matrices exactly coincide.

What about when our complex number isn't on the unit circle?

Problem 40. Choose a new $z = a + bi$ anywhere in the complex plane.

- a) Determine θ , the angle that z makes with the positive x-axis.
- b) Determine r , the magnitude of z .
- c) Come up with expressions for a and b in terms of r and θ (as well involving any other trig functions you need).

Remark. We have, incidentally, just passed from the familiar Cartesian “rectangular” coordinate system to what’s known as the “polar” coordinate system.

- d) Rewrite your matrix from problem 38 in terms of your newfound expressions.
- e) Compare to your matrix from theorem 4. What do you notice? You may also wish to refer back to problem 28.

Theorem 7. *Your observations after having thoughtfully completed problem 40 should almost certainly be encapsulated in a theorem. There are a number of ways you might choose to frame it. We’ll let you do all the heavy lifting.*

♪♪♪ *Fin* ♪♪♪