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Author(s): Blake E. Peterson
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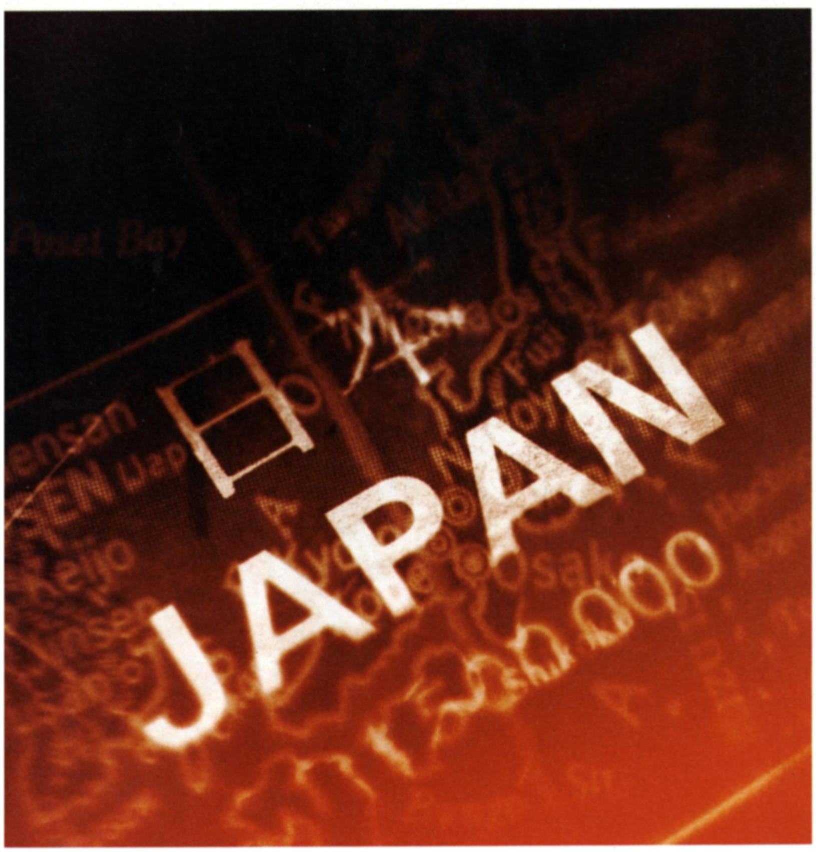
LINEAR AND QUADRATIC CHANGE: *A Problem from Japan*

Blake E. Peterson



In the fall of 2003, I had the opportunity to conduct some research on the student teaching process in Japan. During my seven weeks of research at the junior high school affiliated with Ehime University in Matsuyama, Japan, I observed mathematics lessons taught by student teachers as well as many more lessons taught by experienced teachers. The basis for most of these lessons was wonderfully rich mathematics problems. In these lessons a problem was posed to students, time was given for them to explore it, and then a discussion of the solutions to the problem took place. A detailed description of similar problem-based lessons can be found in *The Teaching Gap* (Stigler and Hiebert 1999) and *The Open-Ended Approach: A New Proposal for Teaching Mathematics* (Becker and Shimada 1997).

Some of the assets of these problems were the connections students were able to make and the variety of representations they were able to employ in solving them. For example, connections were made between tabular, graphical, and symbolic representations, between linear and quadratic equations, and between geometric behavior and algebraic functions. Over time, I began to realize that the richness of these problems had a great deal to do with the connections and representations that were such a prominent part of these lessons.

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Preliminary question: In the figure, as the step changes, ________ also changes.

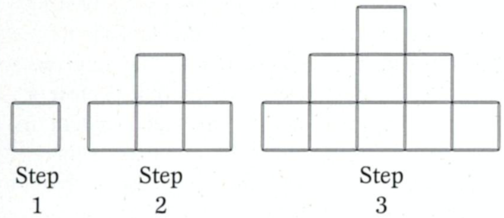

Fig. 1 What attributes change as the step increases?

Many teachers in the United States are making efforts to incorporate the Process Standards from the *Principles and Standards for School Mathematics* (NCTM 2000) into the teaching and learning of mathematics in their classrooms. If students are only presented with routine exercises that focus on a narrowly defined skill, connections are difficult to make. Solving broad, open-ended problems, however, allows students to see connections as part of the problem-solving process. Open-ended problem-solving situations also afford students the opportunity to use various representations as they solve the problem and communicate their solution to their peers.

Rich problems, like the ones I observed being used in Japan, are excellent sources about which to build lessons that incorporate the Process Standards of problem solving, connections, and representations. In this article, I begin by introducing a favorite from among the problems that I saw used by my colleagues in Japan and then go on to describe how this problem plays out in the classroom. Common student approaches to the problem will be presented, along with a discussion of where it might fit in the curriculum.

This problem is centered on the numerical and graphical behavior of linear and quadratic functions and is designed to encourage students to use tables and graphs on their way to describing geometric behavior symbolically. This problem asks students to describe the change in attributes of a sequence of geometric figures. Some attributes change linearly, and others change in a quadratic pattern. The conversations about the differences between linear and quadratic behavior allow students to make many of the types of connections described above.

LINEAR AND NONLINEAR GROWTH

This problem, which I saw used in a ninth-grade mathematics classroom in Japan, seems most appropriate for an algebra 2 class in the United States. Because it generates some linear and some quadratic solutions, it would be a nice problem for students after they have become familiar with both linear and quadratic equations. However, I used it in an algebra 2 class just before students began their study of quadratic equations and found it to be excellent for revisiting the concept of linear functions and also for motivating a discussion about nonlinear (quadratic) functions.

If this problem is introduced prior to a study of quadratic equations, students initially only see cases as linear and not linear. They are unsure what type of equation to use to describe the observed nonlinear situation. Because the nonlinear examples from this problem are quadratic, when I refer to nonlinear behavior, quadratic is implied.

Since most textbooks present students with equations and ask them to create a table and a graph from the equation, they think about linear and quadratic equations as first-degree equations with one variable and second-degree equations with one variable, respectively. Students may also visualize the graph of a line or a parabola, but they make connections between these representations and their corresponding tables far less often. In the problem presented here, students describe a geometric pattern by first building a table of values, then constructing the graph of those values, and finishing with an equation. This order, which differs from that of most textbook problems, allows students to make the connection between the table and other representations more readily. They can also compare the tabular values generated by a linear equation with those generated by a quadratic equation.

The following list gives some examples of responses to the preliminary question (see **fig. 1**) from students in both the Japanese ninth-grade classroom and the United States algebra 2 classroom:

perimeter
height
width
size of enclosing rectangle
number of "toothpicks"
number of interior toothpicks
number of intersections
number of corners
number of convex corners
number of squares
number of nonadjacent squares
number of right angles
sum of the interior angles
number of diagonals
leftover space
number of segments
number of parallel lines
length of longest line
number of rectangles

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As students generate a list of answers, questions of clarification need to be posed, such as "What do you mean by 'toothpicks'?" "What do you mean by 'number of squares'? Do you mean squares of any size?" "Are the diagonals only across one small square or can they cross multiple squares?" "Do intersections include points where two toothpicks meet or just where four toothpicks meet?" Each student or group of students may have their own answers to these questions, but it becomes their responsibility to define specifically what they are considering.

Once a list of changing attributes is identified, students are asked to describe the change in one attribute:

> **Problem:** Using a table, a graph, and an equation, describe the step-by-step change observed in **figure 1**.

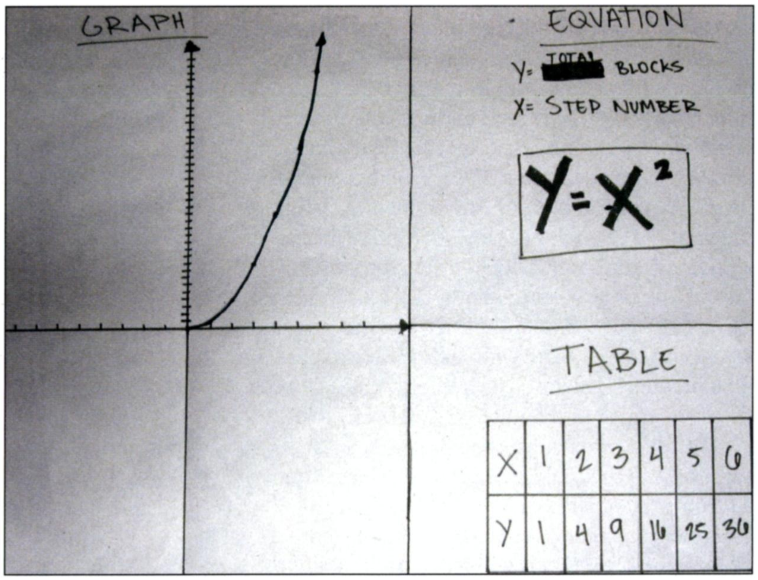

Fig. 2 One group counted the number of blocks (area).

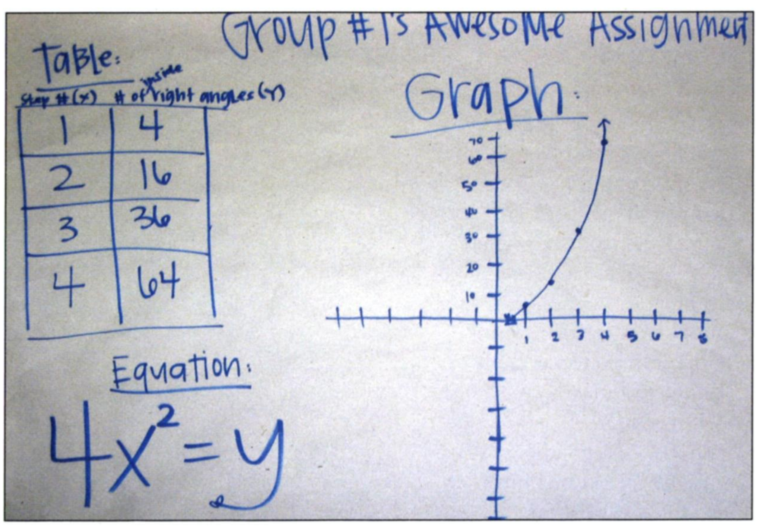

Fig. 3 The number of "inside" right angles is four times the number of blocks.

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The solutions to this problem tend to fall into two general categories: linear and quadratic. As tables are constructed, the students are quickly able to identify patterns that are linear and patterns that are not linear. We will first look at some nonlinear (quadratic) examples and then some linear examples.

Quadratic

Many of the attributes in the list above generate quadratic growth. Those considered by students in the United States include total blocks or area, the number of inside right angles, leftover space, and the number of toothpicks.

Total blocks or area: The total number of blocks, or 1×1 squares, is the same as the area of the figure. One group of students saw this attribute as the total blocks and created the table, graph, and equation shown in **figure 2**. In this example, generating a table of values highlights a pattern that is readily recognized as perfect squares (see lower right-hand corner of **fig. 2**). Thus, the equation likely comes from the ability to recognize a numerical pattern rather than from any knowledge of quadratic functions.

Notice that although the problem situation has a domain of natural numbers, this group constructed a graph that is continuous with a domain of positive real numbers. All other groups, including groups from Japan, made the same generalization. My Japanese colleagues and I chose not to pursue this distinction, but it could easily become a rich point of discussion.

The number of inside right angles: If the number of right angles is investigated, the first question that must be resolved is whether to count the right angles on the outside of the figures. One group of students chose to consider only the interior right angles. Since each 1×1 square in the figure has four right angles in it, the total number of inside right angles is four times as big as the number of 1×1 squares.

The group that chose this approach has all three representations clearly displayed, as can be seen in **figure 3**. Based on observations, I believe that the students were able to generate their equation by looking at the numerical pattern in the table. It is interesting to note that these students had not been formally introduced to quadratic equations at the time they were presented with this problem; thus, it is unlikely that the quadratic equations were generated from a knowledge of the behavior of quadratic functions.

The leftover space: Leftover space is determined by constructing a rectangle around each figure and computing the number of squares in the rectangle that are not part of the original figure. Although the change in this attribute is modeled by a qua-

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dratic function, the students are able to focus on the geometry of the problem in order to generate an equation without any formal knowledge of these nonlinear functions. The student work in **figure 4** highlights how the geometry of this attribute can be quite readily understood.

In the previous cases of the total number of blocks and the number of right angles, the function was easily generated by observing the numerical pattern in the table, but in this case the numerical pattern is more difficult to identify. The geometric pattern, on the other hand, can shed a great deal of light on the function. In the student work (**fig. 4**), the leftover space is rearranged to form a rectangle. This can be seen more clearly in **figure 5**.

As the step increases, the leftover space is in the shape of two inverted staircases. Each staircase height and width increases by one as the step increases. The staircases are placed together as shown in **figure 5** to form a rectangle. Thus, the leftover space for step 2 forms a 1×2 rectangle; the leftover space for step 3 can be rearranged to form a 2×3 rectangle; and the leftover space for step 4 can be rearranged into a 3×4 rectangle. The equation for the area of this new rectangle becomes the function that describes this attribute. Thus, the equation of the area of the leftover space is $A = x(x - 1)$ where x is the step number. After using geometry to generate the equation, connections can be made between the quadratic function and the numerical pattern in the table by verifying that the equation does, in fact, generate the values in the table.

The number of toothpicks: If each of the original figures is viewed as being constructed with toothpicks, then generating a function that counts the number of toothpicks for a given step can be an interesting and challenging problem. In this case, the students created a table (see **fig. 6**) and quickly recognized that it was not linear because the differences between the values in the sequence 4, 13, 26, 43, 64, . . . are not constant. The pattern, however, is difficult to identify numerically as well as geometrically, and the students were unsure of the type of function that could describe the pattern. Since a linear function would have constant differences between successive terms, they concluded that the function must have some type of x^2 term in it. From that point, they used a guess-and-test method to create a function that would match the inputs and outputs in the table. In the guess-and-test process, they had a $2x^2$ term and changed it to a $3x^2$ term, only to realize that the function grew too quickly. Therefore, they went back to the $2x^2$ term and added an x term to make their equation of the form $y = 2x^2 + x + 1$. Next they used guess-and-test approach on the coefficient of the x and the constant term in order to find the appropriate equation of $2x^2 + 3x - 1$.

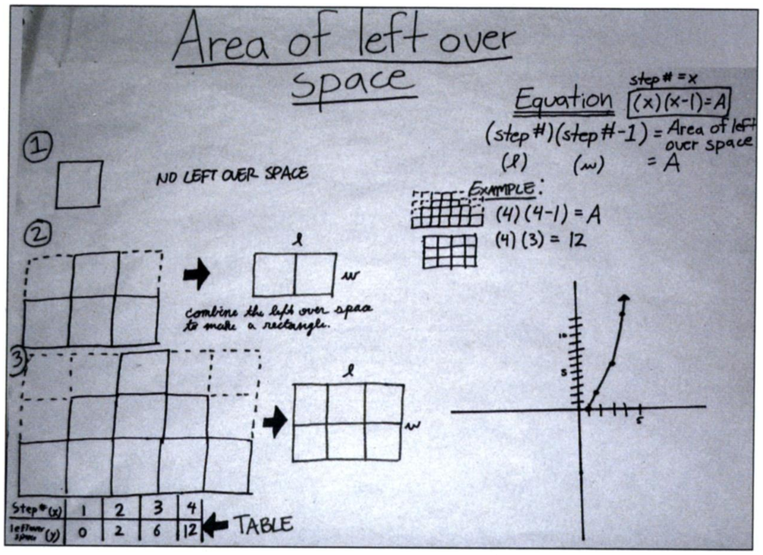

Fig. 4 "Leftover space" requires a slightly more sophisticated approach.

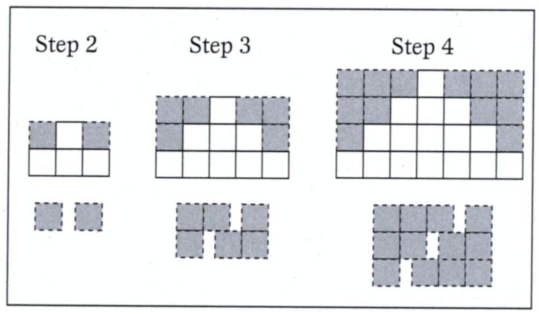

Fig. 5 Leftover space can be arranged to form rectangles.

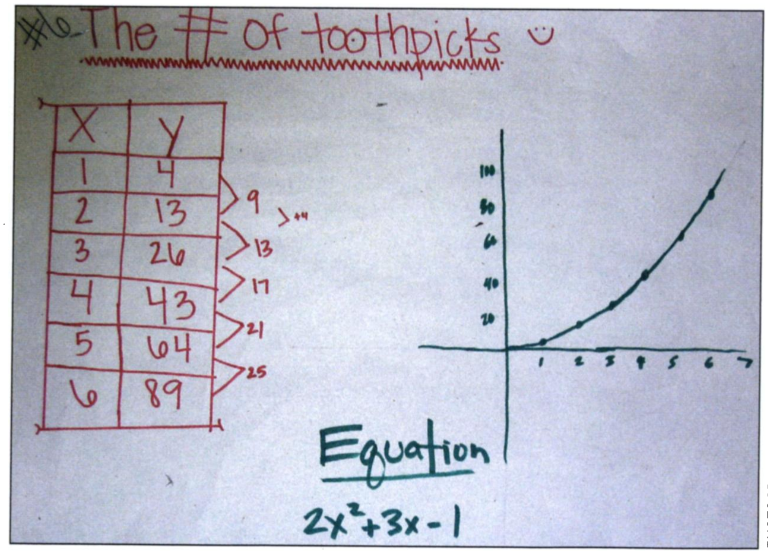

Fig. 6 This group used guess and check to arrive at a quadratic equation.

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A slightly more sophisticated approach to this problem would be to count systematically the number of vertical toothpicks and the number of horizontal toothpicks. Some students began variations on this approach but struggled to put all of

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Table 1

Counting Toothpicks

Step	Vertical Toothpicks	Horizontal Toothpicks	Total
1	$1+1$	$1+1$	4
2	$1+2+2+1$	$1+3+3$	13
3	$1+2+3+3+2+1$	$1+3+5+5$	26
4	$1+2+3+4+4+3+2+1$ $=2(1+2+3+4)$	$1+3+5+7+7$ $=(1+3+5+7)+7$	43
n	$2(1+2+3+\cdots+n)$	$(1+3+5+\cdots+(2n-1))$ $+(2n-1)$	

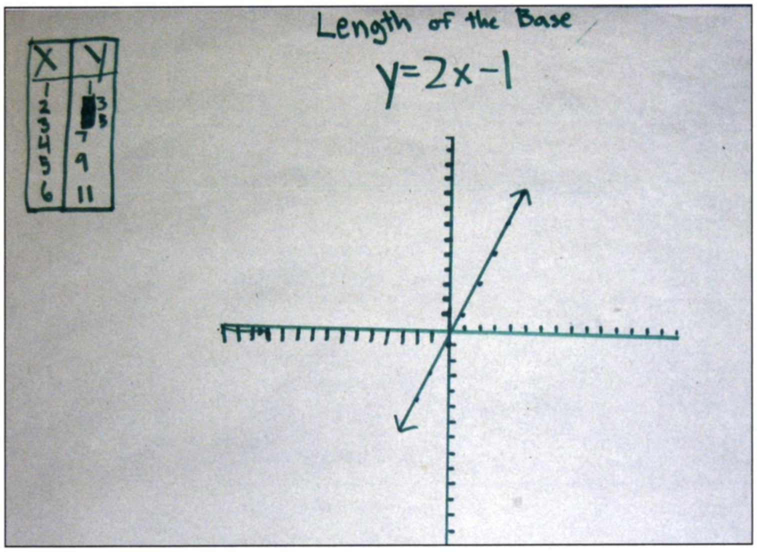

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Fig. 7 The length of the base is a linear function of step number.

the pieces together in the time allotted in order to find the resulting equation. For clarification of the solution to this problem, a summary of this counting method is shown in **table 1**. In the table, the vertical columns are counted left to right, and the horizontal rows are counted top to bottom.

Since the sum of the first n integers is

$$\frac{n(n+1)}{2},$$

the number of vertical toothpicks becomes

$$2\left(\frac{n(n+1)}{2}\right)=n(n+1)=n^2+n.$$

It is also known that the sum of the first n odd integers is n^2, so the number of horizontal toothpicks is $n^2 + 2n - 1$. Adding the number of vertical and horizontal toothpicks yields $(n^2+n)+(n^2+2n-1)=2n^2+3n-1$.

Summary: In each case that generated a quadratic equation, students constructed a table by looking at the geometric figure. They quickly recognized that the pattern in the table was not linear because the differences between consecutive outputs were not constant. Since they had not yet been introduced to quadratic equations, they did not know what to call the pattern other than "not linear." In the case of the total number of blocks and total right angles, perfect squares could be recognized in the table of values, and an equation followed. In the case of the leftover space, numerical patterns were not easily seen, but a pattern in the geometry of the shapes led to an equation. In the final case of the number of toothpicks, neither of the previous methods proved to be productive. Thus, students hypothesized that the equation contained a term to the second degree and used guess and test from that point.

In every case, students started with a physical situation and created a table on their way to writing an equation. Since this order is different from what is usually found in textbooks, it should help students move more flexibly, in either direction, between different representations.

Linear

The first of the two linear examples was generated directly from looking at the length of the base. The second example evolved from a group of students struggling to describe the leftover space because it was not linear; instead, they described the change in the leftover space.

Length of the base: When considering how the length of the base changes as the step changes, the table, graph, and equation shown in **figure 7** are generated. Students can clearly see from the table of values that the y values are increasing by 2 each time the x values increase by 1. Recognizing that the constant differences are indicative of the slope of a line is a valuable connection. This recognizable linear behavior implies that the slope of the equation is 2. The corresponding equation, $y = 2x - 1$, easily follows.

Change of the leftover space: In the quadratic examples, one of the groups had considered the leftover space (see **fig. 4**). A second group also investigated the leftover space and, in their notes, generated the table shown in **figure 8**. When the students looked at the leftover space (0, 2, 6, 12, 20, 30, . . .) and differences in the leftover space (2, 4, 6, 8 ,10, . . .), they realized that the leftover space was not changing linearly because the differences were not constant. When they looked at the difference of the differences (2, 2, 2, 2, . . .), however, they noticed a constant change, indicating a linear pattern. Thus, they created their poster (see **fig. 9**) to describe the behavior of the change in the leftover space instead of the behavior of the left-

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Step	Leftover Space		
1	0		
		+2	
2	2		+2
		+4	
3	6		+2
		+6	
4	12		+2
		+8	
5	20		+2
		+10	
6	30		

Fig. 8 Second differences are constant when computing leftover space.

over space. In the poster, they consolidated their table to show only the step and the change in the leftover space, which would be described by the linear equation $y = 2x - 2$.

In the discussion of these solutions with students, a natural progression could be to start the group of linear solutions and then continue with the quadratics. The last case, the change in the leftover space, however, follows better after a discussion of the leftover space. For this reason, I reversed the order I might typically follow in a classroom.

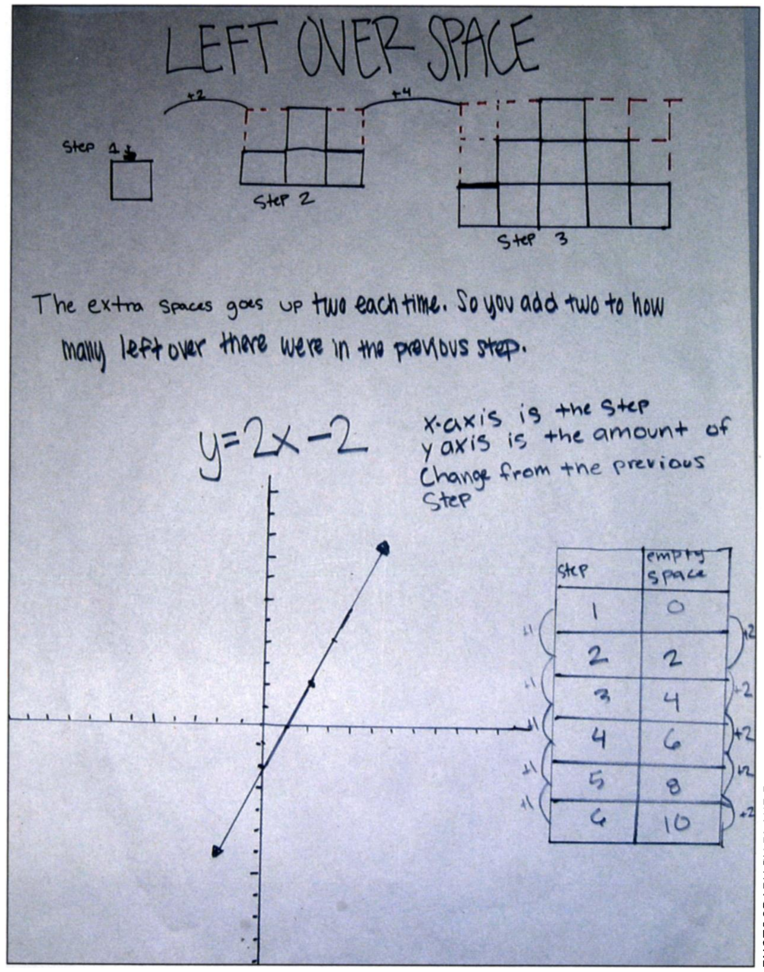

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Fig. 9 The linear function for change in leftover space

A JAPANESE COMPARISON

In Japan, I saw this problem taught in a ninth-grade classroom of forty students working in groups of four. The class worked on this problem for two and a half 50-minute periods. The students spent the first day deciding which changing attribute to investigate and formalizing their thinking. By the end of the first day, a few of the ten groups had presented their solutions. The remainder of the solutions were presented on the second day of the lesson. On the beginning of the third day, the teacher, Ms. Sunada, discussed some generalizations about linear and quadratic functions.

The Japanese student solutions were similar to those of the U.S. students and fell into the same two categories of linear and nonlinear functions. Japanese students also looked at the total number of triangles created by cutting each square in half and at the total number of triangles in the leftover space. The triangle ideas came from a suggestion made by a student early in the discussion and got the whole class thinking about triangles. The reasoning needed to investigate these situations, however, is the same as what U.S. students had to use. Some of the other attributes the Japanese students investigated were the number of triangles with no exterior edge, the maximum number of nonadjacent squares, and the sum of interior angles around the perimeter of the polygon that was formed by the squares.

As the Japanese students presented their solutions, they were encouraged by the teacher to share how they found the equation. Ms. Sunada was very careful to push them to justify how each component of their equation related to the figure and numerical patterns. For example, when one group of students looked at the sum of the interior angles and generated the equation $y = 360 + 720(x - 1)$, she asked them why it was $x - 1$ instead of just x. The discussion of student solutions also focused on similarities and differences between the solutions. Ms. Sunada organized the presentations of the solutions so these similarities and differences would become more obvious.

CONCLUSION

The problem presented here is centered on the representations of tables, graphs, and equations. In addition, the problem was placed in a geometric context, which is yet another representation. The observation of patterns in the geometric situations and the subsequent conversion of these patterns to graphs and equations is also fertile ground for students to make connections.

For the purpose of making connections, "problem selection is especially important because students are unlikely to learn to make connections

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unless they are working on problems or situations that have the potential for suggesting such linkages" (NCTM 2000, p. 359). The problem presented here is particularly nice because the connections occur naturally in the problem-solving process, allowing students to make them without being told precisely what to look for. In Japan and the United States, the attributes that the students selected, the representations they used, and the connections they made were all similar. The learning that occurred was not an artifact of the language or the culture; it was a product of the rich mathematical problem in which they all engaged.

ACKNOWLEDGMENT

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BLAKE E. PETERSON, Peterson@mathed.byu.edu, is an associate professor of mathematics education at Brigham Young University, Provo, UT 84602. His research interests are cooperating teacher/student dialogue and Japanese mathematics teacher education.

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