Assets and Pitfalls to Using Technology in Teaching and Learning Functions

Function is a core concept, a unifying organizer and powerful tool in modern mathematics. Technology pervades modern society. This paper looks at issues arising when the two meet, describing some of the advantages and some of the cautions of using interactive dynamic technology in the study of functions. Some perspectives from the literature help set the stage for the discussion and an example illustrates how technology can be used to explore core function concepts.

Worldwide, a focus on the use of technology as a tool in the teaching of functions is seen in didactical research articles and in classroom practice. Curriculum materials in many countries include graphing and analyzing functions using handheld technology; researchers (for example Lagrange and Artigue, 2009; Hazzan and Goldenberg, 1997; Thompson, 1994) have studied students' interaction with technology and the effect on their understanding of function.

Using Technology in Teaching Functions

Example 1. Lagrange and Artigue (2009) talk of "the global challenge of learning about functions at upper secondary level." Thompson (1994) describes problems students may have conceptualizing a question such as: "A rectangle is situated in the triangle as shown in Figure 1. Find the dimensions of the rectangle that maximize its area."

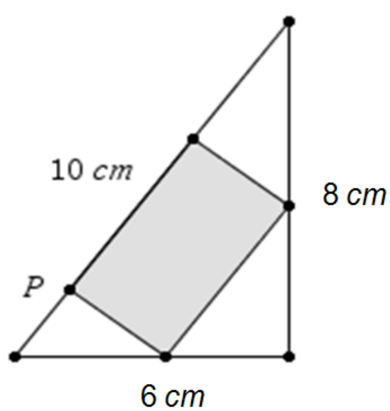


Figure 1: Rectangle inscribed in a triangle

Technology can help with Thompson's issues of conceptualization. Using the ideas of Lagrange and Artigue (2009) for a designed digital environment, a student would be able to explore and see the variation inherent in the situation by dragging point *P*, which is constrained to lie on the triangle's hypotenuse. Dragging point *P* along the hypotenuse changes the shape and hence the rectangle's length, width and area. To explore quantitative changes, the student may use tools in the digital environment to measure the area and the length or width. Doing this, the student observes how the quantities vary as *P* moves to different positions on the triangle's hypotenuse. Figure 2 shows some of the stages of this dragging process.

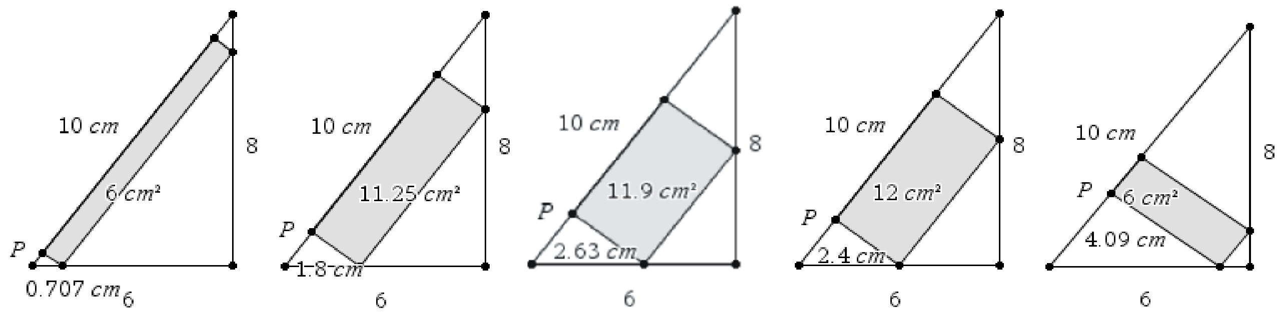


Figure 1: Stages seen in dragging point *P*

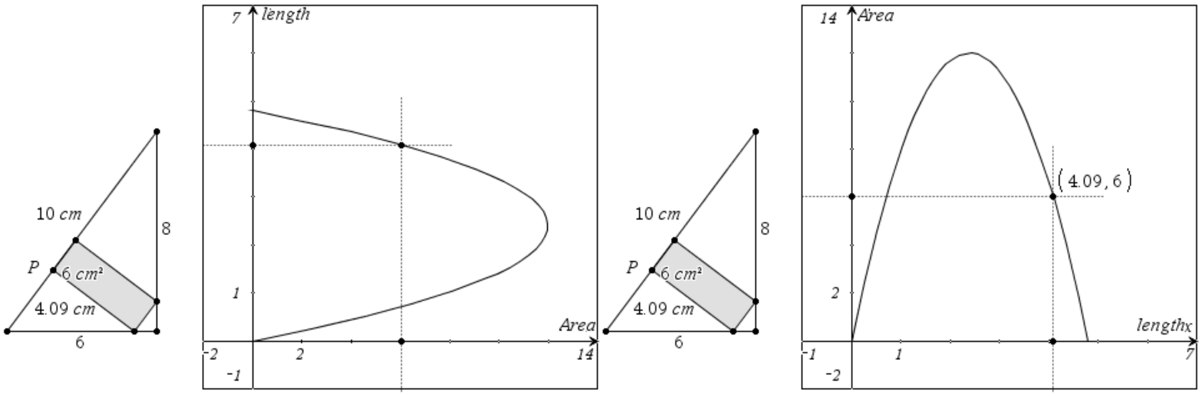
Readers unfamiliar with digital environments should understand that dragging moves only point *P* and the vertices of the rectangle. Thus the observable changes in such a situation are the moving position of point *P*, the shape of the rectangle and the digits in the numbers displaying the measures of the rectangle's variable length and area. The triangle does not move.

Developing the concept of variability with technology

Observing such a transformation on the rectangle as in Figure 2 can be a critical step in the development of the function concept. For Lagrange and Artigue (2009) observing the interaction between the area and length can be a precursor to drawing graphs of the observed variability and to conceiving interactive geometric situations as a pre-algebraic metaphor for a formula. Thompson (1994) says "What has changed because of technological advances are the kind of experiences we can engender in the hope that students eventually create functions as objects" and "we can increase their chances of success by giving explicit attention to imagery as an important aspect of pedagogy and curriculum".

Technology and an approach to the concept of function

For the learner, the process of graphing variability can involve a developing appreciation of the nature of dependence as the variability occurs. Digital environments allow the learner to experiment with the choice of axes on which to represent the dynamism of the experience. In designed digital environments Thompson's problematic triangle of Figure 1 becomes an opportunity for insight into the principles that lie at the heart of the function concept. Figure 3 shows the situation when the measures of area and length are graphed with both choices of axes portrayed.



(a) Variability as relation (b) Variability as function

Figure 3: Thompson's rectangle problem with resulting variability graphs

Transferring the measurements for area and length and plotting a point at the intersection of the lines for each measure's value allows the variation to be represented by the locus of a point moving on the coordinate axes but constrained by the values produced as *P* moves.

Technology and the independent variable

The two graphical representations in Figure 3 provide immediate visual information about two critical aspects of function: domain and range, and correspondence. Figure 3a reveals that the area measure is unsuitable as an independent variable because the input Area = 6 corresponds to two values of length which reveals that area does not specify a unique value of length. Figure 3b, however, demonstrates that any value for length does specify a unique value for area, giving a cogent illustration that each *length* value corresponds to a unique *Area* value. In addition to this insight on correspondence, the constraints on the measured values can be determined from the graphical representation, $0 \le length \le \frac{24}{5}$ and $0 \le Area \le 12$, inviting the student into a reflective discussion about the role of these two sets of numbers.

Developing the function concept with technology

No procedural algebra was needed by the student in the construction of the graphs digitally. In an article about the role of such designed digital environments, Goldenberg and Hazzan (1997) write, "the developers of these software tools provided an unusually rich environment in which students may expand their notions of function and may develop and explore a variety of related ideas. The many differences between algebraic (or graphical or tabular) and dynamic-geometric representations of function provide opportunities for alternative experiences that can, with appropriate reflection, greatly expand students' ideas of function." For Lagrange and Artigue (2009) these explorations in the essentially geometric environments are followed by numerical observations as students trace graphs and browse tables. Students next work algebraically to express domains and a relational formula, progress to algebraic transformations and choose appropriate functional forms as a representation for variation and correspondence. To consolidate the concept of function, students may work with families of functions; the role of parameters may be explored; and students may interpret the results of their mathematics and work on proof.

Example 2. The next example shows another stage in the model proposed by Lagrange and Artigue (2009). But before the problem is presented, a typical pre-requisite problem is posed.

> Begin with square *ABCD* and diagonal segment *BD* in Figure 4. Place a movable point *M*, on the diagonal and form rectangle *ANMP*. Rectangle *ANMP* has a fixed perimeter as you move *M* along the diagonal. The graph represents *BM* on the *x*-axis and the area of rectangle *ANMP* on the *y*-axis. This shows that with a fixed perimeter, the graph of all possible areas formed as a function of the length BM form a parabola.

One asset of this method is that students have a visual model that can be physically moved. Students drag the point *M* along the diagonal and see the change in area of the rectangle as the movement takes place. As *M* is moved, the coordinates of point *K* are displayed so students constantly see the distance *BM* and the area of the rectangle as an ordered pair. Students realize this is an infinite set of points that forms the graph of a continuous function.

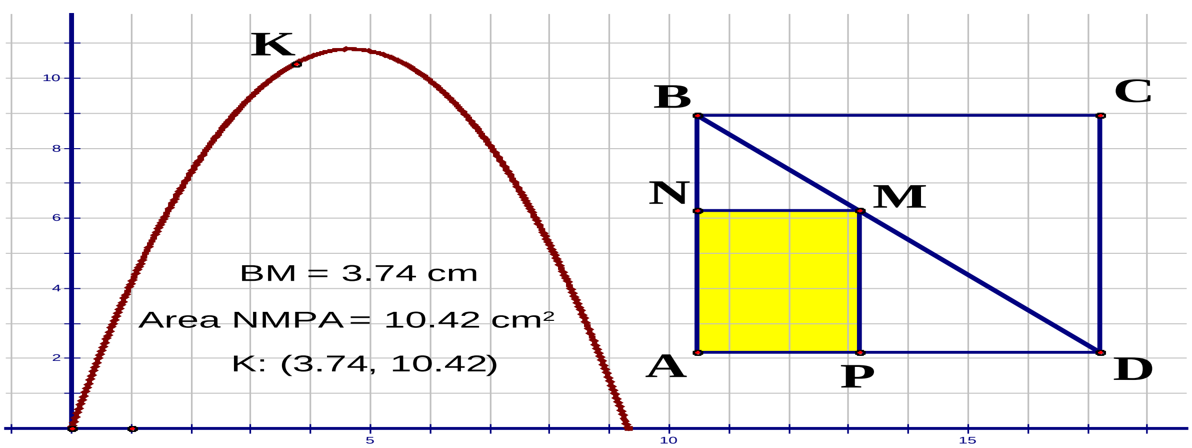


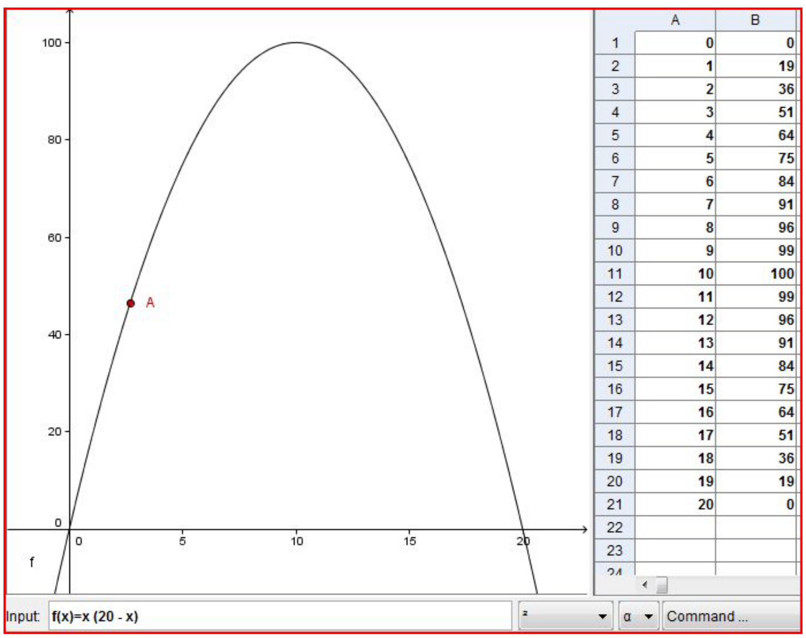
Figure 4: Rectangle *ABCD* with moveable point *M* and resulting graph of area of *ANMP* versus length *BM*

Using a dynamic geometry environment (DGE), students can explore or teachers can model for students' relationships that are functional in nature. A common problem given to students is the following used to demonstrate the final process proposed by Lagrange and Artigue (2009).

What is the maximum area of the rectangle that you can enclose using 40-m of fencing?

To solve this problem, most students will draw a rectangle and label the length *x* and the width, $20 - x$. They know that area equals length times width and develop the equation, $y = f(x) = x(20 - x)$. Using a DGE with a spreadsheet view in the split window mode students graph the function and trace a moveable point along the function exporting these points to the spreadsheet as seen in Figure 5. They see that the area of the rectangle in the problem takes on many different values along a parabola as the length changes, and the spreadsheet allows them to see the values as they move the point along the parabola. The area increases and then decreases showing the students that a maximum value occurs between 0 and 20. Now they must find the maximum value of the area. One approach follows:

The function is a quadratic with its property of symmetry, with the axis of symmetry occurring midway between the two zeros, 0 and 20. Students deduce that the maximum must occur when the length is 10 m. Therefore the area is 100 m^2.



	A	B
1	0	0
2	1	19
3	2	36
4	3	51
5	4	64
6	5	75
7	6	84
8	7	91
9	8	96
10	9	99
11	10	100
12	11	99
13	12	96
14	13	91
15	14	84
16	15	75
17	16	64
18	17	51
19	18	36
20	19	19
21	20	0
22		
23		

Figure 5: Graphical and spreadsheet approach to fencing problem

One asset of this approach is to tie this problem back to the more general prerequisite example above. This affords students a more complete understanding of the problem and how it ties to functions in general by looking at the similarities in the graphs and the models used.

Some Pitfalls in the Use of Technology in the Teaching of Functions

Some mathematicians around the world believe there is too much focus on graphical representations of functions. They believe that this focus presents a problem for the motivation of the teaching and learning of the mathematical structure of the concept of function, and that students can survive using “recipes” on how to plot a given problem with the computer. With this focus, students may refer only to the graphical representation of the function when they are engaged in problem solving and dialogues about functions. This way of thinking can reduce the range and depth of the kind of problems with which students can work. And we are most certainly dealing with a singular view on functions (the graphical representation) that Thompson (1994) criticized.

Others are concerned that students - of all ages - are using calculators in inexpedient ways, e.g., when calculating or calculating $f(2)$ when $f(x) = 2x + 1$. They worry that when students routinely use the calculator to find values of functions at some exact points or to display the graph, it will be very hard for them to understand the meaning of symbolism such as “$f(x + h) = 2f(x)$” when $f(x) = ba^x$ or of an expression like “$f(f(x))$”.

Some worry about the technical issues involved with the mathematics of the technology, for example, the computation $sin(0) = 4.12E\text{-}19$ or SQRT(3.9999999999999999) = 2. This can lead students to solve equations or explore graphs of functions only in the range “given” by the technology. And yet another worrisome example connected to graphing is the apparent “continuity” of $f(x) = \tan(x)$ on some technology. Overall, the general concern here is that many students are not aware of the fact that the limitations of the technology can be mathematically misleading.

If policy makers see technology as supporting instruction, it can be proposed that students using it either can solve more difficult problems or at least more problems of a difficult nature than ones that are not. If policy makers see technology as a goal, then questions related to situations where there are obvious, clear advantages in using technology should be posed. In the first case, there should be clear references to the use of technology in curriculum discussion. In the second case, it should be kept in mind that the entire educational system (including textbooks, teachers, students, and socioeconomics) has to “play along”. The complexity one finds in dealing with this issue is illustrated by the fact that in some countries Computer Algebra System (CAS) usage is accentuated in the curriculum as well as in the “mind setting signals” pushed forward by ministries of education, but the national exams can still be taken without the use of CAS.

Aspects of using technology in the teaching of functions are also related to issues in cognitive psychology. Technology can unintentionally push forward an *instrumental understanding* (Skemp, 1986) of the concept of function; “When I do this, it does this”. One reason this is problematic is that it is difficult to expand the conceptual world of a student with such an understanding – with clear negative implications for the student’s further exploration and consolidation of mathematical concepts. This could be seen as opposed to a *relational understanding* where the concept is understood from the point of view of its relations to other mathematical (and outside mathematics) concepts.

Relational understanding makes it possible to connect the concept of functions to new subjects, methods and ways of reasoning with the already learned material.

The important question is: Is working with functions easier using technology?
On the one hand: If you already know what you are doing, then technology can make it easier to do the work. And technology can, in the hand of a skilled teacher, be used to show properties of "hard to grasp" mathematical concepts such as the concept of functions. *On the other hand*: Mathematics is not engineering. The use of technology can push the "calculate" /"produce numbers results" interpretation of doing mathematics forward instead of the investigation of functional properties via analysis or axiomatic-deductive theory building.

Conclusion

Technology has a variety of meanings when it is related to mathematics education. A scientific calculator and a CAS have very different potentials in relation to the teaching of functions. Given the complexity and the variety of the technology that can be used in mathematics, one has to be very specific in the discussion of roles, possibilities, assets and pitfalls.

Introducing technology into the mathematics curriculum may be an expansion of the syllabus. The policy makers' focus on technology in the classroom can in fact be seen as an expansion of the syllabus content – instead of a means of making the mathematics easier. The basic questions could be "is technology to become an 'aid and tool competency' to replace mathematics" or "is the learning of technology so time-consuming that if it is used in mathematics education, it is a major 'curricula issue'"? There are no easy answers to these questions, but even so around the world; the usage of technology is ever increasing in mathematics education and the teaching of functions. As an international group, we see the value of the use of technology but also see many obstacles along the path to the usage that supports the learning of function.

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