Day 2: Let The Good × Roll

Opener

1. We're going to start with doing the same thing, over and over. The *Monica sequence* is one of the least famous sequences of all time. It starts with 2, then 2 again, then each new term is *twice* the one before it, plus *three times* the one before that. A more formal definition is

$$M(0) = 2$$
$$M(1) = 2$$
$$M(n) = 2 \cdot M(n-1) + 3 \cdot M(n-2) \quad \text{if } n > 1$$

For example, $M(2) = 2 \cdot M(1) + 3 \cdot M(0)$, then $M(3) = 2 \cdot M(2) + 3 \cdot M(1)$, then . . . Use the values of $M(0)$ and $M(1)$ to find $M(2)$, then use the values . . . then lather, rinse, and . . .

a. For the Monica sequence, determine $M(0)$ through $M(8)$.

b. Your table will be given four new pairs of starting numbers. For each pair, determine the first nine numbers (including the two givens). Notice anything?

I noticed it was a vague question. I wonder if the questions will always be this vague.

c. Use your work to find a way to calculate $M(10)$ for the original Monica sequence directly without calculating $M(9)$.

d. Describe some similarities between the five sequences your table worked with.

Important Stuff

2. Find a solution to this system of equations:

$$A + B = 2$$
$$3A - B = 2$$

3. Vince gives you a new starting pair for today's Opener: $V = \begin{bmatrix} 0 \\ 40 \end{bmatrix}$. Determine the first nine numbers for this starting pair *super-quick*.

4. Kimberly adds starting pairs P and Q from today's Opener. What happens!

P is pretty normal!

5. Iris's sequence is $2, 5, 8, 11, 14, \ldots$.

This means $I(0)$ should be 2, $I(1)$ should be 5, and $I(221)$ is almost but not quite 666.

a. Describe the pattern of Iris's sequence:
*Each term in Iris's sequence is*____________________.

b. Write a recursive definition for Iris's sequence by filling in these boxes:

$$I(0) = \boxed{}$$
$$I(n) = I(n-1) + \boxed{} \quad \text{if } n > 0$$

While "a number" is a valid answer . . . some answers are better than others. Sorry.

6. Johnson's sequence is $3, 6, 12, 24, 48, \ldots$

a. Describe the pattern of Johnson's sequence:
*Each term in Johnson's sequence is*____________________.

b. Write a recursive definition for Johnson's sequence by filling in these boxes:

While "more than the last one" is a valid answer . . .

$$J(0) = \boxed{}$$
$$J(n) = \boxed{} \quad \text{if } n > 0$$

7. Calculate these.

a. $(5+\sqrt{2}) + (5-\sqrt{2})$
b. $(5+\sqrt{2}) \cdot (5-\sqrt{2})$

8. Find two numbers with the given sum s and product p.

Product P was a terrible cereal.

a. $s = 10, p = 25$
b. $s = 10, p = 24$
c. $s = 10, p = 23$
d. $s = 10, p = 22$
e. $s = 10, p = 21$
f. $s = 10, p = 20$
g. $s = 10, p = 1$
h. $s = 10, p = -1$
i. $s = 10, p = -299$
j. $s = 100, p = 2451$

Neat Stuff

9. Work through Set 1's Opener using the starting pair $F(0) = x$ and $F(1) = y$. What do you notice?

10. Here's a recursive definition for the sequence $0, 1, 3, 6, 10 \ldots$:

'Recursive" doesn't mean "cursive again."

$$s(0) = 0$$
$$s(n) = s(n-1) + n \quad \text{if } n > 0$$

a. Determine $s(9)$.
b. Determine $s(100)$.

11. A "Monica-like" sequence is defined by

$$R(0) = 5$$
$$R(1) = 19$$
$$R(n) = 2R(n-1) + 3R(n-2) \quad \text{if } n > 1$$

Find a closed rule, such as $R(n) = 3^n$, that matches the sequence, then used the closed rule to find $R(10)$.

A *closed rule* is one like $R(n) = 3^n + (-1)^n$. It has no recursion, and it also has no recursion.

12. Find a solution to this system of equations:

$$A + B = 5$$
$$3A - B = 19$$

13. Which Fibonacci numbers are even, and which are odd? Explain why this happens.

Fibonacci numbers are like marching soldiers in The Wizard of Oz. Oh, ee, oh . . . oh, ee, oh . . .

14. Which Fibonacci numbers are multiples of 5?

15. The *Lucas sequence* is like the Fibonacci sequence, except it starts with 2 and 1 instead of 0 and 1:

$L(2) = 3, L(3) = 4, L(4) = 7$. The Lucas sequence lives on the second floor. Please don't read literature about Fibonacci and Lucas, so that you can find and prove results on your own.

$$L(0) = 2$$
$$L(1) = 1$$
$$L(n) = L(n-1) + L(n-2) \quad \text{if } n > 1$$

Find as many relationships as you can between the numbers in the Lucas sequence and the numbers in the Fibonacci sequence. Try to prove them!

16. Ramona's sequence is the sum of the Lucas and Fibonacci sequences.

Ramona said something about starting pairs and indexes and doubling or something.

$$R(n) = L(n) + F(n)$$

Figure out stuff about Ramona's sequence and relationships between the Lucas and Fibonacci sequences.

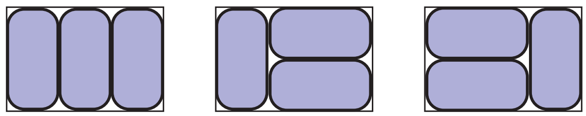
17. Here is the Zucanacci sequence. Figure stuff out!

$$Z(0) = 2i$$
$$Z(1) = 1 + i$$
$$Z(n) = Z(n-1) + Z(n-2) \quad \text{if } n > 1$$

18. Write a recursive rule for $h(n)$ that fits the sequence $1, 10, 44, 160, 536, 1720, 5384 \ldots$

Roy recently learned the ending to Red Hot Chili Peppers songs can be recursive. Give it away now . . .

19. In terms of n, how many ways are there to tile a 2-by-n rectangle with identical 1-by-2 dominoes? Consider any rotations or reflections to be *different* tilings: there are 3 tilings for the 2-by-3 rectangle. Why look, here they are!!



20. In terms of n, how many binary sequences of length n do not have consecutive zeros?

A *binary sequence* is made up of all ones and zeros. For $n = 2$ there are four binary sequences: 00, 01, 10, and 11.

21. Without a calculator, determine the units (ones) digit of $F(100)$ and of $F(1000)$.

22. Describe what happens with the sequence defined by

$$r(0) = 1, \qquad r(n) = 1 + \frac{1}{r(n-1)} \quad \text{if } n > 0$$

23. Some pairs of Fibonacci numbers $F(a)$ and $F(b)$ have common factors. Investigate and find something interesting.

Don't count common factor 1. Well, maybe you can . . .

Tough Stuff

24. Describe a rule you could use to determine, given any integer $n > 1$, which Fibonacci numbers are divisible by n.

25. Prove that any positive integer can be written *in exactly one way* as the sum of one or more non-consecutive Fibonacci numbers. For example: $43 = 34 + 8 + 1$ while $43 = 21 + 13 + 5 + 3 + 1$ would be unacceptable.

There are 100 types of people in the world: those who understand the Zeckendorf representation and those who don't.

26. Find x if

$$\sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \ldots}}}}} = 1$$

Starting Pairs:

$N \begin{bmatrix} 1 \\ -5 \end{bmatrix}$ $O \begin{bmatrix} 0 \\ 4 \end{bmatrix}$ $P \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $Q \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

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