Problem Set 1: It's a New Year for Problem Solving! . . .

Welcome to PCMI! We know you'll learn a great deal of mathematics here—maybe some new tricks, maybe some new perspectives on things with which you're already familiar. A few things you should know about how the class is organized:

- **Don't worry about answering all the questions.** If you're answering every question, that's our fault, not yours.
- **Don't worry about getting to a certain problem number.** Some participants have been known to spend the entire session working on one problem (and perhaps a few of its extensions or consequences).
- **Stop and smell the roses.** Getting the correct answer to a question is not a be-all and end-all in this course. How does the question relate to others you've encountered? How do others think about this question?
- **Be excellent to each other.** Believe that you have something to learn from everyone else. Remember that everyone works at a different pace. Give everyone equal opportunity to express themselves. Don't be afraid to ask questions.
- **Teach only if you have to.** You may feel the temptation to teach others in your group. Fight it! We don't mean you should ignore your classmates but give everyone the chance to discover. If you think it's a good time to teach your colleagues about eigenvalues, think again: the problems should lead to the appropriate mathematics rather than requiring it.

PCMI teachers have solved two previously unsolved problems presented in these courses.

Opener

Let's watch . . . *Survivor!*

Math teachers ready . . . *go!*

Clip can be found online at `http://tinyurl.com/surv21`.

Rules of the game. Figure out what you can! These are all two-player games with a winner and loser.

Think about visualizations that make things simpler to understand.

1. 21, all same type. Pick 1, 2, or 3. Last pick wins.

2. 21, all same type. Pick 1, 2, or 3. Last pick *loses*.

Hopefully you have enough cards, chips, or coins to work with. If you need more, grab 'em!

3. 21, all same type. Pick 2 or 3, unless there's 1 left. Last pick wins.

4. 21, all same type. Pick 2, 3, or 5, unless there's 1 left. Last pick wins.

5. 21, all same type. Make up your own rules and share with another group.

What other variations are there?

6. 13 of one type, 8 of another. On your turn, pick one of either type. Last pick wins.

7. 13 of one type, 8 of another. On your turn, pick one of either type *or* one of each type. Last pick wins.

8. 13 of one type, 8 of another. Pick as many as you want *of the same type*. Last pick wins.

9. 13 of one type, 8 of another. Pick as many as you want of the same type *or* the same number of each type. Last pick wins.

For example, you could leave 10 and 5 after your first pick, but not 10 and 6.

10. 9 of one type, 7 of another, 5 of a third type. Pick as many as you want of the same type. Last pick wins.

11. 9 of one type, 7 of another, 5 of a third type. Pick as many as you want of the same type *or* the same number of . . . hm, lots of options now.

12. For the game in problem 9, (1,2) is a "losing position" because the other player can definitely beat you. Make a list of losing positions (a, b) with $a < b$ and figure out what you can.

A longer list makes it more likely you'll find some amazing stuff.

13. Problem 10 suggests a three-dimensional representation. What's it look like?

Problem Set 2: Chess time. Where's Bobby Fischer?

1. A chess king sits three squares below the top right corner of a chessboard. Two players, alternating turns, move the king toward the lower left corner. The one who puts the king in the corner is the winner. If both players play with perfect strategy, who wins, and how? Oh, you'll also need to figure out what perfect strategy means.

When the king gets there, it says "That's me in the corner."

2. A chess rook (castle) sits three squares below the top right corner of a chessboard. Same rules. Who wins, and how?

When the rook gets there, it says "Don't give up til you drink from the silver cup" and then everyone else wonders what that even means.

3. A chess queen sits three squares below the top right corner. Same rules. Who wins, and how?

When the queen gets there, it says "YAASS, queen."

4. Bring your new knowledge to bear on some of the problems you did yesterday. What do you notice?

5. Repeat the king and rook problems. Oh wait, now these pieces are in 3-D, and they start 7 units to the east, 4 units to the north, and 8 units up from the target space.

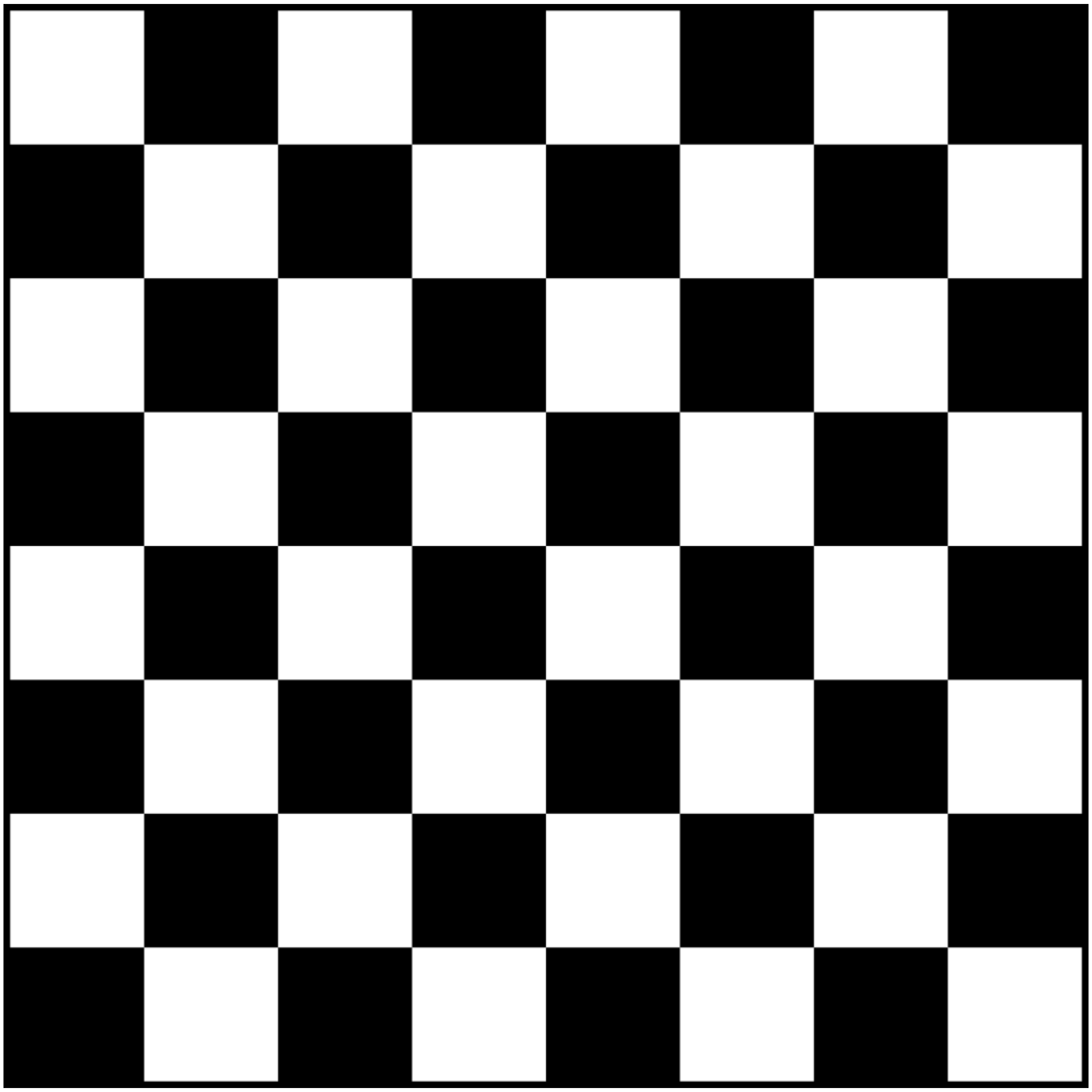
Since you have already done this problem, you surely aren't a rook-ie

6. Repeat the queen problem in fabulous 3-D. You'll have to decide how a queen moves in 3-D.

A queen moves fabulously, of course. But what moves are allowed?

7. **a.** 31, all same type. Pick a power of 3. Last pick wins.
b. 31, all same type. Pick a power of 4. Last pick wins.
c. 31, all same type. Pick a power of 5. Last pick wins.
d. Generalize: what happens when the rules are powers of n?

8. **a.** Show that if n is odd and k is a whole number, then n^k is odd. What are the implications for Problem 7?
b. Show that if n is even and k is a whole number, then n^{2k} is 1 more than a multiple of $(k+1)$, and n^{2k+1} is 1 less than a multiple of $(k+1)$. And this is useful how?



Problem Set 3: Triple Trouble

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A *Pythagorean triple* is three integers (a, b, c) which are side lengths of a right triangle, with hypotenuse length c.

Find a boatload of Pythagorean triples, and classify them into categories.

If you're interested, you can add your triples to this list:

`http://tinyurl.com/la-triples`

It's easy as (a, b, c). It's easy as $(1, 2, 3)$.

No. Sorry, Michael, $(1, 2, 3)$ is not a Pythagorean triple.

1. The value of an integer, mod 4, is its remainder when you divide by 4.

a. What is $3 + 3$ mod 4?

b. What is $3 \cdot 3$ mod 4?

c. Build addition and multiplication tables for mod 4.

These tables are oriented the way addition and multiplication tables should be oriented! Yeah!

$x + y$ mod 4

y \ x	0	1	2	3
3				2
2				
1			3	
0				

$x \cdot y$ mod 4

y \ x	0	1	2	3
3				1
2				
1			2	
0				

2. A *primitive Pythagorean triple* is a Pythagorean triple with no common factors greater than 1.

Primitive Pythagorean triples have discovered the right angle, but not fire.

a. Show that every primitive Pythagorean triple has at least one odd number.

b. Show that in a Pythagorean triple, it isn't possible to have a and b both be odd while c is even.

c. Show that every primitive Pythagorean triple has exactly two odd numbers.

3. Show that every primitive Pythagorean triple must contain a multiple of   .

Oops, forgot the number. Sorry!

4. Find some ways to classify primitive Pythagorean triples, and some ways to generate more of them.

5. Take two odd numbers that differ by two, such as 3 and 5. Then take their reciprocals to make them unit fractions: $\frac{1}{3}$ and $\frac{1}{5}$. Now add them together: $\frac{1}{3} + \frac{1}{5} = \frac{8}{15}$. What do you notice about the numerator and denominator of this sum? Other examples? Generalize? Why does this work?

6. Take any two fractions whose product is 2. Add 2 to each fraction then multiply by the least common denominator to turn into whole numbers. What do you notice? Why does this work?

Note the fractions need not be in simplest reduced form.

7. **a.** Can there be two Pythagorean right triangles with the same perimeter?
b. Fine. Can there be two *primitive* Pythagorean right triangles with the same perimeter?

8. Find *four* different Pythagorean triples with hypotenuse 65, or however many there are. Who knows, there might not even be four.

Whoever wrote this is really lazy. Whoever wrote *this* is also really lazy.

9. Start over, and work the same problems finding *Eisenstein* triples. Eisenstein triples have a 60-degree angle opposite c, but otherwise everything is the same.

10. Work the same problems finding *anti-Eisenstein* triples. Anti-Eisenstein triples have a 120-degree angle opposite c.

Careful, don't put an Eisenstein triple next to an anti-Eisenstein triple! If you do, a plate of antipasto will explode.

11. A *Matsuura triangle* is a triangle for which there is a point inside the triangle forming 120-degree angles with each vertex, and for which all six segments built from this diagram have integer length. Find the smallest-perimeter Matsuura triangle, or prove no such triangle exists.